```
In[1]:= (* Approximate convective terms using TVD schemes *)

(*% Developed and updated by Assoc.Prof.Dr.Eng.Kiril Shterev.
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  % April,4th,2022.
 %
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 % Please cite my papers if you find this information useful:
  %
   % K.Shterev and S.Stefanov,Pressure based finite volume method
 % for calculation of compressible viscous gas flows,Journal of
  % Computational Physics 229 (2010) pp.461-480,doi:10.1016/j.jcp.2009.09.042
  %
  % K.S.Shterev and S.K.Stefanov,A Parallelization of Finite Volume Method
 % for Calculation of Gas Microflows by Domain Decomposition Methods,7th
  % Internnational Conference-Large-ScaleScientific Computations,Sozopol,
Bulgaria,June 04-08,2009. Lecture Notes in Computer Science Volume 5910,
% 2010,DOI:10.1007/978-3-642-12535-5,SJR 0.295.
  %
 % Kiril S.Shterev,GPU implementation of algorithm SIMPLE-TS for calculation
  % of unsteady,viscous,compressible and heat-conductive gas flows,
% URL:https://arxiv.org/abs/1802.04243,2018.
 %
 K.S.Shterev and S.Ivanovska,Comparison of some approximation schemes for
 convective terms for solving gas flow past a square in a microchannel,
APPLICATION OF MATHEMATICS IN TECHNICAL AND NATURAL
 SCIENCES:4th International Conference-AMiTaNS'12,11-16 June 2012,
St.Constantine and Helena,Bulgaria,AIP Conf.Proc.1487,pp.79-87;
doi:http://dx.doi.org/10.1063/1.4758944,ISBN 978-0-7354-1099-2
  %
*)
```

In[3]:=
```
(* Derive numerical equations of partial differencial equations of viscous,
compressible, heat conductive gas for 2D case,
according SIMPLE-TS published in Journal of Computational Physics,
2010, doi:10.1016/j.jcp.2009.09.042 *)
(* The system of PDE equations is:;
```

$$\partial_t(\rho.u)+\partial_x(\rho.u.u)+\partial_y(\rho.v.u)=-A\partial_x p+B(\partial_x(\Gamma\partial_x u)+\partial_y(\Gamma\partial_y u))+\rho.g_x+B\left(\partial_x(\Gamma\partial_x u)+\partial_y(\Gamma\partial_x v)-\frac{2}{3}\partial_x(\Gamma(\partial_x u+\partial_y v))\right)$$

;

$$\partial_t(\rho.v)+\partial_x(\rho.u.v)+\partial_y(\rho.v.v)=-A\partial_y p+B(\partial_x(\Gamma\partial_x v)+\partial_y(\Gamma\partial_y v))+\rho.g_y+B\left(\partial_y(\Gamma\partial_y v)+\partial_x(\Gamma\partial_y u)-\frac{2}{3}\partial_y(\Gamma(\partial_x u+\partial_y v))\right)$$

;

$$\partial_t\rho+\partial_x(\rho.u)+\partial_y(\rho.v)=0$$

;

$$\partial_t(\rho.T)+\partial_x(\rho.u.T)+\partial_y(\rho.v.T)=C_{T1}(\partial_x(\Gamma_\lambda\partial_x T)+\partial_y(\Gamma_\lambda\partial_y T))+C_{T2}.\Gamma.\Phi+C_{T3}.p(\partial_x u+\partial_x v)$$

where:

$$\Phi=2\left((\partial_x u)^2+(\partial_y v)^2\right)+(\partial_x v+\partial_y u)^2-\frac{2}{3}(\partial_x u+\partial_y v)^2$$

*)

In[4]:=
```
(*
TVD scheme for Cartesian grid with constant space step (code C++)
  double TVD(fi1,fi2,fi3,fi4,V)=
     if(fabs(fi3-fi2) < Epsilon)
      {
           if(V > 0.0) return(0.5 * psi_TVD((fi2 - fi1)/(fi3-fi2)) * (fi3-fi2));
           else return(0.5 * psi_TVD((fi4 - fi3)/(fi3-fi2)) * (fi3-fi2));
       }
      else return(0.0);

TVD scheme for Cartesian staggered grid (code C++)
    double TVD(fi1,fi2,fi3,fi4,h1,h2,h3,h4,V)
{
     if(fabs(((fi3)-(fi2))/(0.5*((h2)+(h3))))<Epsilon_TVD)
       {
        if((V)>0.0) return
         (psi_TVD((((fi2)-(fi1))/((fi3)-(fi2)))*(((h2)+(h3))/((h1)+(h2))))*((fi3)-(fi2))*(h2)/((h2)+(h3)));
        else return
         (psi_TVD((((fi4)-(fi3))/((fi3)-(fi2)))*(((h2)+(h3))/((h3)+(h4))))*((fi2)-(fi3))*(h3)/((h2)+(h3)));
       }
       else return(0.0);
   }

   TVD scheme for Cartesian grid
   with constant space step (code C++) - in coefficients
```

```
double TVD_in_coefficients(fi1,fi2,fi3,fi4,V)=
  if(fabs(fi3-fi2) < Epsilon)
   {
        if(V > 0.0) return(0.5 * psi_TVD((fi2 - fi1)/(fi3-fi2)));
        else return(-0.5 * psi_TVD((fi4 - fi3)/(fi3-fi2)));
    }
   else return(0.0);

TVD scheme for Cartesian staggered grid (code C++)
  double TVD_in_coefficients(fi1,fi2,fi3,fi4,h1,h2,h3,h4,V)
{
   if(fabs(((fi3)-(fi2))/(0.5*((h2)+(h3))))<Epsilon_TVD)
     {
      if((V)>0.0) return
       (psi_TVD((((fi2)-(fi1))/((fi3)-(fi2)))*(((h2)+(h3))/((h1)+(h2))))*(h2)/((h2)+(h3)));
      else return(psi_TVD(-(((fi4)-(fi3))/((fi3)-(fi2)))*(((h2)+(h3))/((h3)+(h4))))*(h3)/((h2)+(h3)));
     }
     else return(0.0);
  }

  where
  psi_TVD(r) is TVD scheme
  V - velocity or mass flow rate in approximated point to determine flow direction
  fi1,fi2,fi3,fi4 - neighbour points of approximated value between fi2 and fi3
  h1,h2,h3,h4-space steps of control volume for fi1,fi2,fi3,fi4, respectively
*)
```

In[5]:=
```
(* Integration of equation for u *)
```

In[6]:=
```
(* Integration of unsteady term for u *)
```

$$Iud\rho udt = \frac{hy_{"[j]"}}{2*ht}\left(\left(rho_{"[i\_1j]"}*hx_{"[i\_1]"}+rho_{"[ij]"}*hx_{"[i]"}\right)*u_{"[ij]"}-\left(rhopr_{"[i\_1j]"}*hx_{"[i\_1]"}+rhopr_{"[ij]"}*hx_{"[i]"}\right)*upr_{"[ij]"}\right);$$

In[7]:=
```
(* Integration of convective terms for u *)
```

In[8]:=
```
(*  F1x"[ij]" is defined in point (x_v"[i]", y_v"[j]"), where field variables are defined;
F1x"[ij]" = hy"[j]"*rho"[ij]"*1/2*(u"[i1j]"+u"[ij]") - in new definition,
it is used that rho is defined on Control Surface x_v"[i]";
F1x"[ij]" = 1/2*(Fx"[i1j]"+Fx"[ij]") - old definition  *)
Iudρuudx = Simplify["max(0,F1x"[ij]")" * u"[ij]" - "max(0,-F1x"[ij]")" * u"[i1j]" +
    "F1x[ij]" * "TVD_c_in_coeff(u[i_1j],u[ij],u[i1j],u[i2j],hx[i_1],hx[i],hx[i1],F1x[ij])" *
    (u"[i1j]" - u"[ij]") - ("max(0,F1x"[i_1j]")" * u"[i_1j]" - "max(0,-F1x"[i_1j]")" * u"[ij]" + "F1x[i_1j]" *
      "TVD_c_in_coeff(u[i_2j],u[i_1j],u[ij],u[i1j],hx[i_2],hx[i_1],hx[i],F1x[i_1j])" *
      (u"[ij]" - u"[i_1j]"))];
```

In[9]:=
```
Iudρvudy = Simplify[1/2 * ("max(0,Fy"[i_1j1]")" * u"[ij]" - "max(0,-Fy"[i_1j1]")" * u"[ij1]" + "Fy[i_1j1]" *
      "TVD_s_in_coeff(u[ij_1],u[ij],u[ij1],u[ij2],hy[j_1],hy[j],hy[j1],hy[j2],Fy[i_1j1])" *
      (u"[ij1]" - u"[ij]") + "max(0,Fy"[ij1]")" * u"[ij]" - "max(0,-Fy"[ij1]")" * u"[ij1]" + "Fy[ij1]" *
      "TVD_s_in_coeff(u[ij_1],u[ij],u[ij1],u[ij2],hy[j_1],hy[j],hy[j1],hy[j2],Fy[ij1])" *
      (u"[ij1]" - u"[ij]")
    - ("max(0,Fy"[i_1j]")" * u"[ij_1]" - "max(0,-Fy"[i_1j]")" * u"[ij]" + "Fy[i_1j]" *
        "TVD_s_in_coeff(u[ij_2],u[ij_1],u[ij],u[ij1],hy[j_2],hy[j_1],hy[j],hy[j1],Fy[i_1j])" *
        (u"[ij]" - u"[ij_1]") + "max(0,Fy"[ij]")" * u"[ij_1]" - "max(0,-Fy"[ij]")" * u"[ij]" + "Fy[ij]" *
        "TVD_s_in_coeff(u[ij_2],u[ij_1],u[ij],u[ij1],hy[j_2],hy[j_1],hy[j],hy[j1],Fy[ij])" *
        (u"[ij]" - u"[ij_1]")))]
```

Out[9]=
```
1/2 ((max(0,-Fy[i_1j]) + max(0,Fy[i_1j1]) + max(0,-Fy[ij]) + max(0,Fy[ij1]) - Fy[i_1j1]
      TVD_s_in_coeff(u[ij_1],u[ij],u[ij1],u[ij2],hy[j_1],hy[j],hy[j1],hy[j2],Fy[i_1j1]) -
    Fy[ij1] TVD_s_in_coeff(u[ij_1],u[ij],u[ij1],u[ij2],hy[j_1],hy[j],hy[j1],hy[j2],Fy[ij1]) -
    Fy[i_1j]
      TVD_s_in_coeff(u[ij_2],u[ij_1],u[ij],u[ij1],hy[j_2],hy[j_1],hy[j],hy[j1],Fy[i_1j]) -
    Fy[ij] TVD_s_in_coeff(u[ij_2],u[ij_1],u[ij],u[ij1],hy[j_2],hy[j_1],hy[j],hy[j1],Fy[ij]))
  u[ij] - (max(0,Fy[i_1j]) + max(0,Fy[ij]) - Fy[i_1j]
      TVD_s_in_coeff(u[ij_2],u[ij_1],u[ij],u[ij1],hy[j_2],hy[j_1],hy[j],hy[j1],Fy[i_1j]) -
    Fy[ij] TVD_s_in_coeff(u[ij_2],u[ij_1],u[ij],u[ij1],hy[j_2],hy[j_1],hy[j],hy[j1],Fy[ij]))
  u[ij_1] - (max(0,-Fy[i_1j1]) + max(0,-Fy[ij1]) - Fy[i_1j1]
      TVD_s_in_coeff(u[ij_1],u[ij],u[ij1],u[ij2],hy[j_1],hy[j],hy[j1],hy[j2],Fy[i_1j1]) -
    Fy[ij1]
      TVD_s_in_coeff(u[ij_1],u[ij],u[ij1],u[ij2],hy[j_1],hy[j],hy[j1],hy[j2],Fy[ij1])) u[ij1])
```

In[10]:=
```
(* Integration of diffusion terms for u *)
```

In[11]:=
```
(*
Dux"[i1j]"=B*Γ"[ij]"*hy"[j]"/hx"[i]";
Dux"[ij]"=B*Γ"[i_1j]"*hy"[j]"/hx"[i_1]";
*)
IudΓdudx2 = Dux"[i1j]" * (u"[i1j]" - u"[ij]") - Dux"[ij]" * (u"[ij]" - u"[i_1j]");
(* Interpolation of Γ in middle point is:
   Γyf"[ij]"=Hi(Γ"[ij_1]",Γ"[ij]",hy"[j_1]",hy"[j]")
*)
(*
Duy"[ij]"=B*(hx"[i]"*Γyf"[i_1j]"+hx"[i]"*Γyf"[ij]")*1/(hy"[j_1]"+hy"[j]");
Duy"[ij1]"=B*(hx"[i]"*Γyf"[i_1j1]"+hx"[i]"*Γyf"[ij1]")*1/(hy"[j]"+hy"[j1]");
*)
IudΓdudy2 = Duy"[ij1]" * (u"[ij1]" - u"[ij]") - Duy"[ij]" * (u"[ij]" - u"[ij_1]");
```

In[13]:=
```
(* Integration of pressure term *)
Iudpdx = -A * (p"[ij]" - p"[i_1j]") * hy"[j]";
```

In[14]:=
```
(* Integration of source term *)
IudΓdvdydx = B * ((Γyf"[i_1j1]" * hx"[i_1]" + Γyf"[ij1]" * hx"[i]") / (hx"[i_1]" + hx"[i]") * (v"[ij1]" - v"[i_1j1]") -
     (Γyf"[i_1j]" * hx"[i_1]" + Γyf"[ij]" * hx"[i]") / (hx"[i_1]" + hx"[i]") * (v"[ij]" - v"[i_1j]"));
IudΓdvdxdy = B * (Γ"[ij]" * (v"[ij1]" - v"[ij]") - Γ"[i_1j]" * (v"[i_1j1]" - v"[i_1j]"));
```

In[16]:=
```
(* The source term is: Su = B(∂x(Γ∂xu)+∂y(Γ∂xv)-2/3∂x(Γ(∂xu+∂yv))) *)
IuSu = IudΓdudx2 + IudΓdvdydx - 2/3 * (IudΓdudx2 + IudΓdvdxdy)
```

Out[16]=

$$Dux_{[i1j]}\left(u_{[i1j]}-u_{[ij]}\right)-Dux_{[ij]}\left(-u_{[i\_1j]}+u_{[ij]}\right)-\frac{2}{3}\left(Dux_{[i1j]}\left(u_{[i1j]}-u_{[ij]}\right)-Dux_{[ij]}\left(-u_{[i\_1j]}+u_{[ij]}\right)+B\left(-\left(\left(-v_{[i\_1j]}+v_{[i\_1j1]}\right)\Gamma_{[i\_1j]}\right)+\left(-v_{[ij]}+v_{[ij1]}\right)\Gamma_{[ij]}\right)\right)+B\left(-\frac{\left(-v_{[i\_1j]}+v_{[ij]}\right)\left(hx_{[i\_1]}\,\Gamma yf_{[i\_1j]}+hx_{[i]}\,\Gamma yf_{[ij]}\right)}{hx_{[i]}+hx_{[i\_1]}}+\frac{\left(-v_{[i\_1j1]}+v_{[ij1]}\right)\left(hx_{[i\_1]}\,\Gamma yf_{[i\_1j1]}+hx_{[i]}\,\Gamma yf_{[ij1]}\right)}{hx_{[i]}+hx_{[i\_1]}}\right)$$

In[17]:=
```
(* Derive numerical coefficients for source term *)
```

In[18]:=
```
aSu0 = Simplify[-Coefficient[IuSu, u"[ij]"]]
```

Out[18]=

$$\frac{1}{3}\left(Dux_{[i1j]}+Dux_{[ij]}\right)$$

In[19]:=
```
aSu1 = Simplify[Coefficient[IuSu, u"[i_1j]"]]
```

Out[19]=

$$\frac{Dux_{[ij]}}{3}$$

In[20]:= aSu2 = Simplify[Coefficient[IuSu, u"[i1j]"]]

Out[20]= $\frac{Dux_{[i1j]}}{3}$

In[21]:= aSu3 = Simplify[Coefficient[IuSu, u"[ij_1]"]]

Out[21]= 0

In[22]:= aSu4 = Simplify[Coefficient[IuSu, u"[ij1]"]]

Out[22]= 0

In[23]:= Suc =
Simplify[-(IuSu - (aSu0 * u"[ij]" - (aSu1 * u"[i_1j]" + aSu2 * u"[i1j]" + aSu3 * u"[ij_1]" + aSu4 * u"[ij1]")))]

Out[23]= $\frac{1}{3\,(hx_{[i]} + hx_{[i\_1]})}\Big(-2\,Dux_{[ij]}\,(hx_{[i]} + hx_{[i\_1]})\,(u_{[i\_1j]} - u_{[ij]}) - 2\,Dux_{[i1j]}\,(hx_{[i]} + hx_{[i\_1]})\,(u_{[i1j]} - u_{[ij]}) + B\,hx_{[i\_1]}\,\big(-2\,v_{[ij]}\,\Gamma_{[ij]} + 2\,v_{[ij1]}\,\Gamma_{[ij]} + v_{[i\_1j]}\,(2\,\Gamma_{[i\_1j]} - 3\,\Gamma yf_{[i\_1j]}) + 3\,v_{[ij]}\,\Gamma yf_{[i\_1j]} - 3\,v_{[ij1]}\,\Gamma yf_{[i\_1j1]} + v_{[i\_1j1]}\,(-2\,\Gamma_{[i\_1j]} + 3\,\Gamma yf_{[i\_1j1]})\big) + B\,hx_{[i]}\,\big(-2\,v_{[ij]}\,\Gamma_{[ij]} + 2\,v_{[ij1]}\,\Gamma_{[ij]} + v_{[i\_1j]}\,(2\,\Gamma_{[i\_1j]} - 3\,\Gamma yf_{[ij]}) + 3\,v_{[ij]}\,\Gamma yf_{[ij]} - 3\,v_{[ij1]}\,\Gamma yf_{[ij1]} + v_{[i\_1j1]}\,(-2\,\Gamma_{[i\_1j]} + 3\,\Gamma yf_{[ij1]})\big)\Big)$

In[24]:= (* Check the derived numerical coefficients - the result have to be zero: *)
Simplify[IuSu - (aSu0 * u"[ij]" - (aSu1 * u"[i_1j]" + aSu2 * u"[i1j]" + aSu3 * u"[ij_1]" + aSu4 * u"[ij1]" + Suc))]

Out[24]= 0

In[25]:= (**)

In[26]:=
```
(* All tesms are moved to the left hand
 side to derive the numerical coefficients. *)
uExpresion =
 FullSimplify[(IudρudT + Iudρuudx + Iudρvudy + Iudpdx - (IudΓdudx2 + IudΓdudy2)) - IuSu]
```

Out[26]=
$$\frac{1}{6}\Big(6\,A\,hy_{[j]}\,(p_{[i\_1j]}-p_{[ij]}) - 6\,(\max(0,F1x_{[i\_1j]}) - F1x[i\_1j]\,\text{TVD\_c\_in\_coeff}(u[i\_2j],u[i\_1j],u[ij],u[i1j],hx[i\_2],hx[i\_1],hx[i],F1x[i\_1j]))\,u_{[i\_1j]} - 6\,(\max(0,-F1x_{[ij]}) - F1x[ij]\,\text{TVD\_c\_in\_coeff}(u[i\_1j],u[ij],u[i1j],u[i2j],hx[i\_1],hx[i],hx[i1],F1x[ij]))\,u_{[i1j]} + 6\,(\max(0,-F1x_{[i\_1j]}) + \max(0,F1x_{[ij]}) - F1x[ij]\,\text{TVD\_c\_in\_coeff}(u[i\_1j],u[ij],u[i1j],u[i2j],hx[i\_1],hx[i],hx[i1],F1x[ij]) - F1x[i\_1j]\,\text{TVD\_c\_in\_coeff}(u[i\_2j],u[i\_1j],u[ij],u[i1j],hx[i\_2],hx[i\_1],hx[i],F1x[i\_1j]))\,u_{[ij]} + 3\,(\max(0,-Fy_{[i\_1j]}) + \max(0,Fy_{[i\_1j1]}) + \max(0,-Fy_{[ij]}) + \max(0,Fy_{[ij1]}) - Fy[i\_1j1]\,\text{TVD\_s\_in\_coeff}(u[ij\_1],u[ij],u[ij1],u[ij2],hy[j\_1],hy[j],hy[j1],hy[j2],Fy[i\_1j1]) - Fy[ij1]\,\text{TVD\_s\_in\_coeff}(u[ij\_1],u[ij],u[ij1],u[ij2],hy[j\_1],hy[j],hy[j1],hy[j2],Fy[ij1]) - Fy[i\_1j]\,\text{TVD\_s\_in\_coeff}(u[ij\_2],u[ij\_1],u[ij],u[ij1],hy[j\_2],hy[j\_1],hy[j],hy[j1],Fy[i\_1j]) - Fy[ij]\,\text{TVD\_s\_in\_coeff}(u[ij\_2],u[ij\_1],u[ij],u[ij1],hy[j\_2],hy[j\_1],hy[j],hy[j1],Fy[ij]))\,u_{[ij]} + \frac{3\,hy_{[j]}\,(hx_{[i\_1]}\,rho_{[i\_1j]} + hx_{[i]}\,rho_{[ij]})\,u_{[ij]}}{ht} + 8\,Dux_{[ij]}\,(-u_{[i\_1j]}+u_{[ij]}) + 8\,Dux_{[i1j]}\,(-u_{[i1j]}+u_{[ij]}) + 6\,Duy_{[ij]}\,(u_{[ij]}-u_{[ij\_1]}) - 3\,(\max(0,Fy_{[i\_1j]}) + \max(0,Fy_{[ij]}) - Fy[i\_1j]\,\text{TVD\_s\_in\_coeff}(u[ij\_2],u[ij\_1],u[ij],u[ij1],hy[j\_2],hy[j\_1],hy[j],hy[j1],Fy[i\_1j]) - Fy[ij]\,\text{TVD\_s\_in\_coeff}(u[ij\_2],u[ij\_1],u[ij],u[ij1],hy[j\_2],hy[j\_1],hy[j],hy[j1],Fy[ij]))\,u_{[ij\_1]} + 6\,Duy_{[ij1]}\,(u_{[ij]}-u_{[ij1]}) - 3\,(\max(0,-Fy_{[i\_1j1]}) + \max(0,-Fy_{[ij1]}) - Fy[i\_1j1]\,\text{TVD\_s\_in\_coeff}(u[ij\_1],u[ij],u[ij1],u[ij2],hy[j\_1],hy[j],hy[j1],hy[j2],Fy[i\_1j1]) - Fy[ij1]\,\text{TVD\_s\_in\_coeff}(u[ij\_1],u[ij],u[ij1],u[ij2],hy[j\_1],hy[j],hy[j1],hy[j2],Fy[ij1]))\,u_{[ij1]} - \frac{3\,hy_{[j]}\,(hx_{[i\_1]}\,rhopr_{[i\_1j]} + hx_{[i]}\,rhopr_{[ij]})\,upr_{[ij]}}{ht} + 4\,B\,(v_{[i\_1j]}-v_{[i\_1j1]})\,\Gamma_{[i\_1j]} + 4\,B\,(-v_{[ij]}+v_{[ij1]})\,\Gamma_{[ij]} + \frac{6\,B\,(-v_{[i\_1j]}+v_{[ij]})\,(hx_{[i\_1]}\,\Gamma yf_{[i\_1j]} + hx_{[i]}\,\Gamma yf_{[ij]})}{hx_{[i]}+hx_{[i\_1]}} + \frac{6\,B\,(v_{[i\_1j1]}-v_{[ij1]})\,(hx_{[i\_1]}\,\Gamma yf_{[i\_1j1]} + hx_{[i]}\,\Gamma yf_{[ij1]})}{hx_{[i]}+hx_{[i\_1]}}\Big)$$

```
In[27]:= (* Derive numerical coefficients *)
au0 = Simplify[Coefficient[uExpresion, u"[ij]"]]

Out[27]= 1/6 (8 Dux[i1j] + 8 Dux[ij] + 3 (2 max(0, -F1x[i_1j]) + 2 max(0, F1x[ij]) +
    max(0, -Fy[i_1j]) + max(0, Fy[i_1j1]) + max(0, -Fy[ij]) + max(0, Fy[ij1]) - 2 F1x[ij]
      TVD_c_in_coeff(u[i_1j], u[ij], u[i1j], u[i2j], hx[i_1], hx[i], hx[i1], F1x[ij]) - 2 F1x[i_1j]
      TVD_c_in_coeff(u[i_2j], u[i_1j], u[ij], u[i1j], hx[i_2], hx[i_1], hx[i], F1x[i_1j]) - Fy[i_1j1]
      TVD_s_in_coeff(u[ij_1], u[ij], u[ij1], u[ij2], hy[j_1], hy[j], hy[j1], hy[j2], Fy[i_1j1]) -
    Fy[ij1] TVD_s_in_coeff(u[ij_1], u[ij], u[ij1], u[ij2], hy[j_1], hy[j], hy[j1], hy[j2], Fy[ij1]) -
    Fy[i_1j]
      TVD_s_in_coeff(u[ij_2], u[ij_1], u[ij], u[ij1], hy[j_2], hy[j_1], hy[j], hy[j1], Fy[i_1j]) -
    Fy[ij] TVD_s_in_coeff(u[ij_2], u[ij_1], u[ij], u[ij1], hy[j_2], hy[j_1], hy[j], hy[j1], Fy[ij]) +
    2 Duy[ij] + 2 Duy[ij1] + (hx[i_1] hy[j] rho[i_1j])/ht + (hx[i] hy[j] rho[ij])/ht))

In[28]:= au1 = Simplify[-Coefficient[uExpresion, u"[i_1j]"]]

Out[28]= max(0, F1x[i_1j]) -
  F1x[i_1j] TVD_c_in_coeff(u[i_2j], u[i_1j], u[ij], u[i1j], hx[i_2], hx[i_1], hx[i], F1x[i_1j]) + (4 Dux[ij])/3

In[29]:= au2 = Simplify[-Coefficient[uExpresion, u"[i1j]"]]

Out[29]= max(0, -F1x[ij]) -
  F1x[ij] TVD_c_in_coeff(u[i_1j], u[ij], u[i1j], u[i2j], hx[i_1], hx[i], hx[i1], F1x[ij]) + (4 Dux[i1j])/3

In[30]:= au3 = Simplify[-Coefficient[uExpresion, u"[ij_1]"]]

Out[30]= 1/2 (max(0, Fy[i_1j]) + max(0, Fy[ij]) -
    Fy[i_1j] TVD_s_in_coeff(u[ij_2], u[ij_1], u[ij], u[ij1], hy[j_2], hy[j_1], hy[j], hy[j1], Fy[i_1j]) -
    Fy[ij] TVD_s_in_coeff(u[ij_2], u[ij_1], u[ij], u[ij1], hy[j_2], hy[j_1], hy[j], hy[j1], Fy[ij]) +
    2 Duy[ij])

In[31]:= au4 = Simplify[-Coefficient[uExpresion, u"[ij1]"]]

Out[31]= 1/2 (max(0, -Fy[i_1j1]) + max(0, -Fy[ij1]) -
    Fy[i_1j1] TVD_s_in_coeff(u[ij_1], u[ij], u[ij1], u[ij2], hy[j_1], hy[j], hy[j1], hy[j2], Fy[i_1j1]) -
    Fy[ij1] TVD_s_in_coeff(u[ij_1], u[ij], u[ij1], u[ij2], hy[j_1], hy[j], hy[j1], hy[j2], Fy[ij1]) +
    2 Duy[ij1])
```

In[32]:=
```
bu =
  Simplify[-(uExpresion - (au0 * u"[ij]" - (au1 * u"[i_1j]" + au2 * u"[i1j]" + au3 * u"[ij_1]" + au4 * u"[ij1]")))]
```

Out[32]=

$$\frac{1}{6\,ht\,(hx_{[i]} + hx_{[i\_1]})}\Big(3\,hx_{[i]}^2\,hy_{[j]}\,rhopr_{[ij]}\,upr_{[ij]} + hx_{[i\_1]}\big(hy_{[j]}(-6\,A\,ht\,p_{[i\_1j]} + 6\,A\,ht\,p_{[ij]} + 3\,hx_{[i\_1]}\,rhopr_{[i\_1j]}\,upr_{[ij]}) + 2\,B\,ht\,(2\,v_{[ij]}\,\Gamma_{[ij]} - 2\,v_{[ij1]}\,\Gamma_{[ij]} - 3\,v_{[ij]}\,\Gamma yf_{[i\_1j]} + v_{[i\_1j]}(-2\,\Gamma_{[i\_1j]} + 3\,\Gamma yf_{[i\_1j]}) + v_{[i\_1j1]}(2\,\Gamma_{[i\_1j]} - 3\,\Gamma yf_{[i\_1j1]}) + 3\,v_{[ij1]}\,\Gamma yf_{[i\_1j1]})\big) + hx_{[i]}\big(hy_{[j]}(-6\,A\,ht\,p_{[i\_1j]} + 6\,A\,ht\,p_{[ij]} + 3\,hx_{[i\_1]}(rhopr_{[i\_1j]} + rhopr_{[ij]})\,upr_{[ij]}) + 2\,B\,ht\,(2\,v_{[ij]}\,\Gamma_{[ij]} - 2\,v_{[ij1]}\,\Gamma_{[ij]} - 3\,v_{[ij]}\,\Gamma yf_{[ij]} + v_{[i\_1j]}(-2\,\Gamma_{[i\_1j]} + 3\,\Gamma yf_{[ij]}) + v_{[i\_1j1]}(2\,\Gamma_{[i\_1j]} - 3\,\Gamma yf_{[ij1]}) + 3\,v_{[ij1]}\,\Gamma yf_{[ij1]})\big)\Big)$$

In[33]:=
```
(* Check the derived numerical coefficients - the result has to be zero: *)
Simplify[uExpresion - (au0 * u"[ij]" - (au1 * u"[i_1j]" + au2 * u"[i1j]" + au3 * u"[ij_1]" + au4 * u"[ij1]" + bu))]
```

Out[33]= 0

In[34]:=
```
au0TVD = -"F1x[ij]" "TVD_c_in_coeff(u[i_1j],u[ij],u[i1j],u[i2j],hx[i_1],hx[i],hx[i1],F1x[ij])" -
   "F1x[i_1j]" "TVD_c_in_coeff(u[i_2j],u[i_1j],u[ij],u[i1j],hx[i_2],hx[i_1],hx[i],F1x[i_1j])" -
   1/2 * ("Fy[i_1j1]"
      "TVD_s_in_coeff(u[ij_1],u[ij],u[ij1],u[ij2],hy[j_1],hy[j],hy[j1],hy[j2],Fy[i_1j1])" +
     "Fy[ij1]"
      "TVD_s_in_coeff(u[ij_1],u[ij],u[ij1],u[ij2],hy[j_1],hy[j],hy[j1],hy[j2],Fy[ij1])" +
     "Fy[i_1j]"
      "TVD_s_in_coeff(u[ij_2],u[ij_1],u[ij],u[ij1],hy[j_2],hy[j_1],hy[j],hy[j1],Fy[i_1j])" +
     "Fy[ij]"
      "TVD_s_in_coeff(u[ij_2],u[ij_1],u[ij],u[ij1],hy[j_2],hy[j_1],hy[j],hy[j1],Fy[ij])");
```

In[35]:=
```
au0NoTVD = Simplify[au0 - au0TVD]
```

Out[35]=

$$\frac{1}{6\,ht}\Big(8\,ht\,Dux_{[i1j]} + 8\,ht\,Dux_{[ij]} + 3\big(2\max(0,-F1x_{[i\_1j]})\,ht + 2\max(0,F1x_{[ij]})\,ht + \max(0,-Fy_{[i\_1j]})\,ht + \max(0,Fy_{[i\_1j1]})\,ht + \max(0,-Fy_{[ij]})\,ht + \max(0,Fy_{[ij1]})\,ht + 2\,ht\,Duy_{[ij]} + 2\,ht\,Duy_{[ij1]} + hx_{[i\_1]}\,hy_{[j]}\,rho_{[i\_1j]} + hx_{[i]}\,hy_{[j]}\,rho_{[ij]}\big)\Big)$$

In[36]:=
```
Simplify[uExpresion -
   ((au0NoTVD + au0TVD) * u"[ij]" - (au1 * u"[i_1j]" + au2 * u"[i1j]" + au3 * u"[ij_1]" + au4 * u"[ij1]" + bu))]
```

Out[36]= 0

In[37]:=
```
(**)
```

In[38]:=
```
(* Integration of equation for v *)
```

In[39]:=
```
(* Integration of unsteady term for v *)
Ivdρvdt = hx"[i]"/(2 * ht) ((rho"[ij_1]" * hy"[j_1]" + rho"[ij]" * hy"[j]") * v"[ij]" -
    (rhopr"[ij_1]" * hy"[j_1]" + rhopr"[ij]" * hy"[j]") * vpr"[ij]");
```

In[40]:=
```
(* Integration of convective terms for v *)
```

In[41]:=
```
Ivdρuvdx = Simplify[1/2 ("max(0,Fx"[i1j]")" * v"[ij]" - "max(0,-Fx"[i1j]")" * v"[i1j]" + "Fx[i1j]" *
      "TVD_s_in_coeff(v[i_1j],v[ij],v[i1j],v[i2j],hx[i_1],hx[i],hx[i1],hx[i2],Fx[i1j])" *
      (v"[i1j]" - v"[ij]") + "max(0,Fx"[i1j_1]")" * v"[ij]" - "max(0,-Fx"[i1j_1]")" * v"[i1j]" + "Fx[i1j_1]" *
      "TVD_s_in_coeff(v[i_1j],v[ij],v[i1j],v[i2j],hx[i_1],hx[i],hx[i1],hx[i2],Fx[i1j_1])" *
      (v"[i1j]" - v"[ij]")
     - ("max(0,Fx"[ij]")" * v"[i_1j]" - "max(0,-Fx"[ij]")" * v"[ij]" + "Fx[ij]" *
        "TVD_s_in_coeff(v[i_2j],v[i_1j],v[ij],v[i1j],hx[i_2],hx[i_1],hx[i],hx[i1],Fx[ij])" *
        (v"[ij]" - v"[i_1j]") + "max(0,Fx"[ij_1]")" * v"[i_1j]" - "max(0,-Fx"[ij_1]")" * v"[ij]" + "Fx[ij_1]" *
        "TVD_s_in_coeff(v[i_2j],v[i_1j],v[ij],v[i1j],hx[i_2],hx[i_1],hx[i],hx[i1],Fx[ij_1])"
        * (v"[ij]" - v"[i_1j]")))];
```

In[42]:=
```
(*  F1y"[ij]" is defined in point (x_v"[i]", y_v"[j]"), where field variables are defined;
F1y"[ij]" = hx"[i]"*rho"[ij]"*1/2*(v"[ij1]"+v"[ij]") - new definition,
it is used that rho is defined on Control Surface y_v"[j]";
F1y"[ij]" = 1/2*(Fy"[ij1]"+Fy"[ij]") - old definition  *)
Ivdρvvdy = Simplify["max(0,F1y"[ij]")" * v"[ij]" - "max(0,-F1y"[ij]")" * v"[ij1]" +
    "F1y[ij]" * "TVD_c_in_coeff(v[ij_1],v[ij],v[ij1],v[ij2],hy[j_1],hy[j],hy[j1],F1y[ij])" *
     (v"[ij1]" - v"[ij]") - ("max(0,F1y"[ij_1]")" * v"[ij_1]" - "max(0,-F1y"[ij_1]")" * v"[ij]" + "F1y[ij_1]" *
      "TVD_c_in_coeff(v[ij_2],v[ij_1],v[ij],v[ij1],hy[j_2],hy[j_1],hy[j],F1y[ij_1])" *
      (v"[ij]" - v"[ij_1]"))];
```

In[43]:=
```
(* Integration of diffusion terms for v *)
```

In[44]:=
```
(* Interpolation of Γ in middle point is:
    Γxf"[ij]"=Hi(Γ"[i_1j]",Γ"[ij]",hx"[i_1]",hx"[i]")
*)
(*
Dvx"[ij]"=B*(hy"[j_1]"*Γxf"[ij_1]"+hy"[j]"*Γxf"[ij]")*1/(hx"[i_1]"+hx"[i]");
Dvx"[i1j]"=B*(hy"[j_1]"*Γxf"[i1j_1]"+hy"[j]"*Γxf"[i1j]")*1/(hx"[i]"+hx"[i1]");
*)
IvdΓdvdx2 = Dvx"[i1j]" * (v"[i1j]" - v"[ij]") - Dvx"[ij]" * (v"[ij]" - v"[i_1j]");
```

In[45]:=
```
(*
Dvy"[ij1]"=B*Γ"[ij]"*hx"[i]"/hy"[j]";
Dvy"[ij]"= B*Γ"[ij_1]"*hx"[i]"/hy"[j_1]";
*)
IvdΓdvdy2 = Dvy"[ij1]" * (v"[ij1]" - v"[ij]") - Dvy"[ij]" * (v"[ij]" - v"[ij_1]");
```

In[46]:=
```
(* Integration of mass forces term (the gravity term) *)
Ivρgy = (rho"[ij_1]" * hy"[j_1]" + rho"[ij]" * hy"[j]") * (gy * hx"[i]")/2;
```

In[47]:=
```
(* Integration of pressure term *)
Ivdpdy = -A * (p"[ij]" - p"[ij_1]") * hx"[i]";
```

In[48]:=
```
(* Integration of source term *)
IvdΓdudxdy = B * ((Γxf"[i1j_1]" * hy"[j_1]" + Γxf"[i1j]" * hy"[j]") / (hy"[j_1]" + hy"[j]") * (u"[i1j]" - u"[i1j_1]") -
      (Γxf"[ij_1]" * hy"[j_1]" + Γxf"[ij]" * hy"[j]") / (hy"[j_1]" + hy"[j]") * (u"[ij]" - u"[ij_1]"));
IvdΓdudydx = B * (Γ"[ij]" * (u"[i1j]" - u"[ij]") - Γ"[ij_1]" * (u"[i1j_1]" - u"[ij_1]"));
```

In[50]:=
```
(* The source term is: Sv = B(∂y(Γ∂yv) + ∂x(Γ∂yu) - 2/3 ∂y(Γ(∂xu + ∂yv))) *)
IvSv = IvdΓdvdy2 + IvdΓdudxdy - 2/3 * (IvdΓdudydx + IvdΓdvdy2)
```

Out[50]=

$$-\mathrm{Dvy}_{[ij]}\left(v_{[ij]}-v_{[ij\_1]}\right)+\mathrm{Dvy}_{[ij1]}\left(-v_{[ij]}+v_{[ij1]}\right)-\frac{2}{3}\left(-\mathrm{Dvy}_{[ij]}\left(v_{[ij]}-v_{[ij\_1]}\right)+\mathrm{Dvy}_{[ij1]}\left(-v_{[ij]}+v_{[ij1]}\right)+B\left(\left(u_{[i1j]}-u_{[ij]}\right)\Gamma_{[ij]}-\left(u_{[i1j\_1]}-u_{[ij\_1]}\right)\Gamma_{[ij\_1]}\right)\right)+B\left(\frac{\left(u_{[i1j]}-u_{[i1j\_1]}\right)\left(hy_{[j]}\,\Gamma xf_{[i1j]}+hy_{[j\_1]}\,\Gamma xf_{[i1j\_1]}\right)}{hy_{[j]}+hy_{[j\_1]}}-\frac{\left(u_{[ij]}-u_{[ij\_1]}\right)\left(hy_{[j]}\,\Gamma xf_{[ij]}+hy_{[j\_1]}\,\Gamma xf_{[ij\_1]}\right)}{hy_{[j]}+hy_{[j\_1]}}\right)$$

In[51]:=
```
(* Derive coefficients for source term *)
```

In[52]:=
```
aSv0 = Simplify[-Coefficient[IvSv, v"[ij]"]]
```

Out[52]=

$$\frac{1}{3}\left(\mathrm{Dvy}_{[ij]}+\mathrm{Dvy}_{[ij1]}\right)$$

In[53]:= aSv1 = Simplify[Coefficient[IvSv, v"[i_1j]"]]

Out[53]= 0

In[54]:= aSv2 = Simplify[Coefficient[IvSv, v"[i1j]"]]

Out[54]= 0

In[55]:= aSv3 = Simplify[Coefficient[IvSv, v"[ij_1]"]]

Out[55]= $\frac{Dvy_{[ij]}}{3}$

In[56]:= aSv4 = Simplify[Coefficient[IvSv, v"[ij1]"]]

Out[56]= $\frac{Dvy_{[ij1]}}{3}$

In[57]:= Svc =
Simplify[-(IvSv - (aSv0 * v"[ij]" - (aSv1 * v"[i_1j]" + aSv2 * v"[i1j]" + aSv3 * v"[ij_1]" + aSv4 * v"[ij1]")))]

Out[57]= $\frac{1}{3\,(hy_{[j]} + hy_{[j\_1]})}$

$$(2\,Dvy_{[ij]}\,(hy_{[j]} + hy_{[j\_1]})\,(v_{[ij]} - v_{[ij\_1]}) + 2\,Dvy_{[ij1]}\,(hy_{[j]} + hy_{[j\_1]})\,(v_{[ij]} - v_{[ij1]}) + B\,hy_{[j]}\,(-2\,u_{[i1j\_1]}\,\Gamma_{[ij\_1]} + 2\,u_{[ij\_1]}\,\Gamma_{[ij\_1]} + u_{[i1j]}\,(2\,\Gamma_{[ij]} - 3\,\Gamma xf_{[i1j]}) + 3\,u_{[i1j\_1]}\,\Gamma xf_{[i1j]} - 3\,u_{[ij\_1]}\,\Gamma xf_{[ij]} + u_{[ij]}\,(-2\,\Gamma_{[ij]} + 3\,\Gamma xf_{[ij]})) + B\,hy_{[j\_1]}\,(-2\,u_{[i1j\_1]}\,\Gamma_{[ij\_1]} + 2\,u_{[ij\_1]}\,\Gamma_{[ij\_1]} + u_{[i1j]}\,(2\,\Gamma_{[ij]} - 3\,\Gamma xf_{[i1j\_1]}) + 3\,u_{[i1j\_1]}\,\Gamma xf_{[i1j\_1]} - 3\,u_{[ij\_1]}\,\Gamma xf_{[ij\_1]} + u_{[ij]}\,(-2\,\Gamma_{[ij]} + 3\,\Gamma xf_{[ij\_1]})))$$

In[58]:= (* Check the derived coefficients - the result has to be zero: *)
Simplify[IvSv - (aSv0 * v"[ij]" - (aSv1 * v"[i_1j]" + aSv2 * v"[i1j]" + aSv3 * v"[ij_1]" + aSv4 * v"[ij1]" + Svc))]

Out[58]= 0

In[59]:= (**)

In[60]:=
```
(* All tesms are moved to the left hand
 side to derive the numerical coefficients. *)
vExpresion = FullSimplify[
  (Ivdρvdt + Ivdρuvdx + Ivdρvvdy + Ivdpdy - (IvdΓdvdx2 + IvdΓdvdy2 + Ivρgy)) - IvSv]
```

Out[60]=
```
1/6 (-3 (max(0, Fx[ij]) + max(0, Fx[ij_1]) - Fx[ij]
      TVD_s_in_coeff(v[i_2j], v[i_1j], v[ij], v[i1j], hx[i_2], hx[i_1], hx[i], hx[i1], Fx[ij]) - Fx[ij_1]
      TVD_s_in_coeff(v[i_2j], v[i_1j], v[ij], v[i1j], hx[i_2], hx[i_1], hx[i], hx[i1], Fx[ij_1]) +
     2 Dvx[ij]) v[i_1j] - 3 (max(0, -Fx[i1j]) + max(0, -Fx[i1j_1]) -
     Fx[i1j] TVD_s_in_coeff(v[i_1j], v[ij], v[i1j], v[i2j], hx[i_1], hx[i], hx[i1], hx[i2], Fx[i1j]) -
     Fx[i1j_1]
      TVD_s_in_coeff(v[i_1j], v[ij], v[i1j], v[i2j], hx[i_1], hx[i], hx[i1], hx[i2], Fx[i1j_1]) +
     2 Dvx[i1j]) v[i1j] + 3 (2 max(0, F1y[ij]) + 2 max(0, -F1y[ij_1]) + max(0, Fx[i1j]) +
     max(0, Fx[i1j_1]) + max(0, -Fx[ij]) + max(0, -Fx[ij_1]) -
     2 F1y[ij] TVD_c_in_coeff(v[ij_1], v[ij], v[ij1], v[ij2], hy[j_1], hy[j], hy[j1], F1y[ij]) -
     2 F1y[ij_1] TVD_c_in_coeff(v[ij_2], v[ij_1], v[ij], v[ij1], hy[j_2], hy[j_1], hy[j], F1y[ij_1]) -
     Fx[i1j] TVD_s_in_coeff(v[i_1j], v[ij], v[i1j], v[i2j], hx[i_1], hx[i], hx[i1], hx[i2], Fx[i1j]) -
     Fx[i1j_1]
      TVD_s_in_coeff(v[i_1j], v[ij], v[i1j], v[i2j], hx[i_1], hx[i], hx[i1], hx[i2], Fx[i1j_1])) v[ij] +
  (-3 Fx[ij] TVD_s_in_coeff(v[i_2j], v[i_1j], v[ij], v[i1j], hx[i_2], hx[i_1], hx[i], hx[i1], Fx[ij]) -
     3 Fx[ij_1]
      TVD_s_in_coeff(v[i_2j], v[i_1j], v[ij], v[i1j], hx[i_2], hx[i_1], hx[i], hx[i1], Fx[ij_1]) +
     6 Dvx[i1j] + 6 Dvx[ij] + 8 Dvy[ij] + 8 Dvy[ij1]) v[ij] - 6 (max(0, F1y[ij_1]) -
     F1y[ij_1] TVD_c_in_coeff(v[ij_2], v[ij_1], v[ij], v[ij1], hy[j_2], hy[j_1], hy[j], F1y[ij_1]))
   v[ij_1] - 6 (max(0, -F1y[ij]) - F1y[ij]
      TVD_c_in_coeff(v[ij_1], v[ij], v[ij1], v[ij2], hy[j_1], hy[j], hy[j1], F1y[ij])) v[ij1] -
  8 (Dvy[ij] v[ij_1] + Dvy[ij1] v[ij1]) - 1/ht 3 hx[i] (2 A ht p[ij] - 2 A ht p[ij_1] +
     (hy[j] rho[ij] + hy[j_1] rho[ij_1]) (gy ht - v[ij]) + (hy[j] rhopr[ij] + hy[j_1] rhopr[ij_1]) vpr[ij]) +
  4 B ((u[i1j] - u[ij]) Γ[ij] + (-u[i1j_1] + u[ij_1]) Γ[ij_1]) +
  1/(hy[j] + hy[j_1])
  6 B (hy[j] ((-u[i1j] + u[i1j_1]) Γxf[i1j] + (u[ij] - u[ij_1]) Γxf[ij]) +
     hy[j_1] ((-u[i1j] + u[i1j_1]) Γxf[i1j_1] + (u[ij] - u[ij_1]) Γxf[ij_1])))
```

In[61]:=
```
(* Derive numerical coefficients *)
av0 = Simplify[Coefficient[vExpresion, v"[ij]"]]
```

Out[61]=

$$\frac{1}{6}\Big(6\max(0,F1y_{[ij]})+6\max(0,-F1y_{[ij\_1]})+3\max(0,Fx_{[i1j]})+3\max(0,Fx_{[i1j\_1]})+3\max(0,-Fx_{[ij]})+3\max(0,-Fx_{[ij\_1]})-$$
$$6\,\text{F1y[ij] TVD\_c\_in\_coeff(v[ij\_1],v[ij],v[ij1],v[ij2],hy[j\_1],hy[j],hy[j1],F1y[ij])}-$$
$$6\,\text{F1y[ij\_1] TVD\_c\_in\_coeff(v[ij\_2],v[ij\_1],v[ij],v[ij1],hy[j\_2],hy[j\_1],hy[j],F1y[ij\_1])}-$$
$$3\,\text{Fx[i1j] TVD\_s\_in\_coeff(v[i\_1j],v[ij],v[i1j],v[i2j],hx[i\_1],hx[i],hx[i1],hx[i2],Fx[i1j])}-$$
$$3\,\text{Fx[i1j\_1] TVD\_s\_in\_coeff(v[i\_1j],v[ij],v[i1j],v[i2j],hx[i\_1],hx[i],hx[i1],hx[i2],Fx[i1j\_1])}-$$
$$3\,\text{Fx[ij] TVD\_s\_in\_coeff(v[i\_2j],v[i\_1j],v[ij],v[i1j],hx[i\_2],hx[i\_1],hx[i],hx[i1],Fx[ij])}-$$
$$3\,\text{Fx[ij\_1] TVD\_s\_in\_coeff(v[i\_2j],v[i\_1j],v[ij],v[i1j],hx[i\_2],hx[i\_1],hx[i],hx[i1],Fx[ij\_1])}+$$
$$6\,Dvx_{[i1j]}+6\,Dvx_{[ij]}+8\,Dvy_{[ij]}+8\,Dvy_{[ij1]}+\frac{3\,hx_{[i]}\,hy_{[j]}\,rho_{[ij]}}{ht}+\frac{3\,hx_{[i]}\,hy_{[j\_1]}\,rho_{[ij\_1]}}{ht}\Big)$$

In[62]:=
```
av1 = Simplify[-Coefficient[vExpresion, v"[i_1j]"]]
```

Out[62]=

$$\frac{1}{2}\Big(\max(0,Fx_{[ij]})+\max(0,Fx_{[ij\_1]})-$$
$$\text{Fx[ij] TVD\_s\_in\_coeff(v[i\_2j],v[i\_1j],v[ij],v[i1j],hx[i\_2],hx[i\_1],hx[i],hx[i1],Fx[ij])}-$$
$$\text{Fx[ij\_1] TVD\_s\_in\_coeff(v[i\_2j],v[i\_1j],v[ij],v[i1j],hx[i\_2],hx[i\_1],hx[i],hx[i1],Fx[ij\_1])}+$$
$$2\,Dvx_{[ij]}\Big)$$

In[63]:=
```
av2 = Simplify[-Coefficient[vExpresion, v"[i1j]"]]
```

Out[63]=

$$\frac{1}{2}\Big(\max(0,-Fx_{[i1j]})+\max(0,-Fx_{[i1j\_1]})-$$
$$\text{Fx[i1j] TVD\_s\_in\_coeff(v[i\_1j],v[ij],v[i1j],v[i2j],hx[i\_1],hx[i],hx[i1],hx[i2],Fx[i1j])}-$$
$$\text{Fx[i1j\_1] TVD\_s\_in\_coeff(v[i\_1j],v[ij],v[i1j],v[i2j],hx[i\_1],hx[i],hx[i1],hx[i2],Fx[i1j\_1])}+$$
$$2\,Dvx_{[i1j]}\Big)$$

In[64]:=
```
av3 = Simplify[-Coefficient[vExpresion, v"[ij_1]"]]
```

Out[64]=

$$\max(0,F1y_{[ij\_1]})-$$
$$\text{F1y[ij\_1] TVD\_c\_in\_coeff(v[ij\_2],v[ij\_1],v[ij],v[ij1],hy[j\_2],hy[j\_1],hy[j],F1y[ij\_1])}+\frac{4\,Dvy_{[ij]}}{3}$$

In[65]:=
```
av4 = Simplify[-Coefficient[vExpresion, v"[ij1]"]]
```

Out[65]=

$$\max(0,-F1y_{[ij]})-$$
$$\text{F1y[ij] TVD\_c\_in\_coeff(v[ij\_1],v[ij],v[ij1],v[ij2],hy[j\_1],hy[j],hy[j1],F1y[ij])}+\frac{4\,Dvy_{[ij1]}}{3}$$

In[66]:=
```
bv =
 Simplify[-(vExpresion - (av0 * v"[ij]" - (av1 * v"[i_1j]" + av2 * v"[i1j]" + av3 * v"[ij_1]" + av4 * v"[ij1]"))))]
```

Out[66]=

$$\frac{1}{2\,ht} hx_{[i]} \left(2\,A\,ht\,p_{[ij]} - 2\,A\,ht\,p_{[ij\_1]} + gy\,ht\,hy_{[j]}\,rho_{[ij]} + gy\,ht\,hy_{[j\_1]}\,rho_{[ij\_1]} + hy_{[j]}\,rhopr_{[ij]}\,vpr_{[ij]} + hy_{[j\_1]}\,rhopr_{[ij\_1]}\,vpr_{[ij]}\right) + \frac{1}{3\left(hy_{[j]} + hy_{[j\_1]}\right)} B \left(hy_{[j]} \left(2\,u_{[i1j\_1]}\,\Gamma_{[ij\_1]} - 2\,u_{[ij\_1]}\,\Gamma_{[ij\_1]} - 3\,u_{[i1j\_1]}\,\Gamma xf_{[i1j]} + u_{[i1j]}\left(-2\,\Gamma_{[ij]} + 3\,\Gamma xf_{[i1j]}\right) + u_{[ij]}\left(2\,\Gamma_{[ij]} - 3\,\Gamma xf_{[ij]}\right) + 3\,u_{[ij\_1]}\,\Gamma xf_{[ij]}\right) + hy_{[j\_1]}\left(2\,u_{[i1j\_1]}\,\Gamma_{[ij\_1]} - 2\,u_{[ij\_1]}\,\Gamma_{[ij\_1]} - 3\,u_{[i1j\_1]}\,\Gamma xf_{[i1j\_1]} + u_{[i1j]}\left(-2\,\Gamma_{[ij]} + 3\,\Gamma xf_{[i1j\_1]}\right) + u_{[ij]}\left(2\,\Gamma_{[ij]} - 3\,\Gamma xf_{[ij\_1]}\right) + 3\,u_{[ij\_1]}\,\Gamma xf_{[ij\_1]}\right)\right)$$

In[67]:=
```
(* Check the derived numerical coefficients - the result has to be zero: *)
Simplify[vExpresion - (av0 * v"[ij]" - (av1 * v"[i_1j]" + av2 * v"[i1j]" + av3 * v"[ij_1]" + av4 * v"[ij1]" + bv))]
```

Out[67]= 0

In[68]:=
```
(**)
```

In[69]:=
```
(* Derive the pressure equation
  The Pressure equation is deduced after integration equation
  for conservation of mass and substitution of velocities. It
  is multiplied to time step. This make algorithm more stable,
when are used small time steps for calculation of supersonic fluid flow.
   Integrated equation for conservation of mass:
   ∂tρ*hx"[i]"*hy"[j]"+(rhou"[i1j]"*u"[i1j]"-rhou"[ij]"*u"[ij]")*hy"[j]"+
    (rhov"[ij1]"*v"[ij1]"-rhov"[ij]"*v"[ij]")*hx"[i]"=0

   Substutude in integrated equation for
   conservation of mass the velocities in using preudo velocities:
    u"[ij]"=upseudo"[ij]"-du"[ij]"*(p"[ij]"-p"[i_1j]")
     v"[ij]"=vpseudo"[ij]"-dv"[ij]"*(p"[ij]"-p"[ij_1]")
 *)
```

In[70]:=
```
(* In unsteady term the density have to be substututed with
  pressure using eqation of state. At this way the numerical equation
  for pressure satisfy the sufficient condition for convergence of
  iterative method and no under relaxation coefficients are needed: *)
Ipdrhodt = Simplify[(p"[ij]"/Temper"[ij]" - ppr"[ij]"/Temperpr"[ij]") * hx"[i]" * hy"[j]"];
```

```
In[71]:= Ipdrhoudx = Simplify[(rhou"[i1j]" * (upseudo"[i1j]" - du"[i1j]" * (p"[i1j]" - p"[ij]")) -
            rhou"[ij]" * (upseudo"[ij]" - du"[ij]" * (p"[ij]" - p"[i_1j]"))) * hy"[j]" * ht];

In[72]:= Ipdrhovdy = Simplify[(rhov"[ij1]" * (vpseudo"[ij1]" - dv"[ij1]" * (p"[ij1]" - p"[ij]")) -
            rhov"[ij]" * (vpseudo"[ij]" - dv"[ij]" * (p"[ij]" - p"[ij_1]"))) * hx"[i]" * ht];

In[73]:= pExpresion = FullSimplify[(Ipdrhodt + Ipdrhoudx + Ipdrhovdy)]
```

Out[73]= $hx_{[i]}\, hy_{[j]} \left(\frac{p_{[ij]}}{Temper_{[ij]}} - \frac{ppr_{[ij]}}{Temperpr_{[ij]}}\right) + ht\, hy_{[j]} \left(rhou_{[i1j]} \left(du_{[i1j]} \left(-p_{[i1j]} + p_{[ij]}\right) + upseudo_{[i1j]}\right) - rhou_{[ij]} \left(du_{[ij]} \left(p_{[i\_1j]} - p_{[ij]}\right) + upseudo_{[ij]}\right)\right) + ht\, hx_{[i]} \left(-rhov_{[ij]} \left(dv_{[ij]} \left(-p_{[ij]} + p_{[ij\_1]}\right) + vpseudo_{[ij]}\right) + rhov_{[ij1]} \left(dv_{[ij1]} \left(p_{[ij]} - p_{[ij1]}\right) + vpseudo_{[ij1]}\right)\right)$

```
In[74]:= (* Derice numerical coefficients *)
         ap0 = Simplify[Coefficient[pExpresion, p"[ij]"]]
```

Out[74]= $ht\, du_{[i1j]}\, hy_{[j]}\, rhou_{[i1j]} + ht\, du_{[ij]}\, hy_{[j]}\, rhou_{[ij]} + hx_{[i]} \left(ht\, dv_{[ij]}\, rhov_{[ij]} + ht\, dv_{[ij1]}\, rhov_{[ij1]} + \frac{hy_{[j]}}{Temper_{[ij]}}\right)$

```
In[75]:= ap1 = Simplify[-Coefficient[pExpresion, p"[i_1j]"]]
```

Out[75]= $ht\, du_{[ij]}\, hy_{[j]}\, rhou_{[ij]}$

```
In[76]:= ap2 = Simplify[-Coefficient[pExpresion, p"[i1j]"]]
```

Out[76]= $ht\, du_{[i1j]}\, hy_{[j]}\, rhou_{[i1j]}$

```
In[77]:= ap3 = Simplify[-Coefficient[pExpresion, p"[ij_1]"]]
```

Out[77]= $ht\, dv_{[ij]}\, hx_{[i]}\, rhov_{[ij]}$

```
In[78]:= ap4 = Simplify[-Coefficient[pExpresion, p"[ij1]"]]
```

Out[78]= $ht\, dv_{[ij1]}\, hx_{[i]}\, rhov_{[ij1]}$

```
In[79]:= bp =
          Simplify[-(pExpresion - (ap0 * p"[ij]" - (ap1 * p"[i_1j]" + ap2 * p"[i1j]" + ap3 * p"[ij_1]" + ap4 * p"[ij1]")))]
```

Out[79]= $ht\, hy_{[j]} \left(-rhou_{[i1j]}\, upseudo_{[i1j]} + rhou_{[ij]}\, upseudo_{[ij]}\right) + hx_{[i]} \left(\frac{hy_{[j]}\, ppr_{[ij]}}{Temperpr_{[ij]}} + ht\, rhov_{[ij]}\, vpseudo_{[ij]} - ht\, rhov_{[ij1]}\, vpseudo_{[ij1]}\right)$

```
In[80]:= (* Check the derived coefficients - the result has to be zero: *)
         Simplify[
          -(pExpresion - (ap0 * p"[ij]" - (ap1 * p"[i_1j]" + ap2 * p"[i1j]" + ap3 * p"[ij_1]" + ap4 * p"[ij1]" + bp)))]
```

Out[80]= 0

In[81]:=
```
(**)
```

In[82]:=
```
(* Derive the energy equation *)
```

In[83]:=
```
(* Integration of unsteady term.
   It is multiplicated by time step to make numerical equation more stable,
when are used small time steps for calculation of supersonic fluid flows. *)
ITdrhoTdt = Simplify[(rho"[ij]" * Temper"[ij]" - rhopr"[ij]" * Temperpr"[ij]") * hx"[i]" * hy"[j]"];
```

In[84]:=
```
(* Integration of convective terms *)
```

In[85]:=
```
ITdrhouTdx =
  Simplify[("max(0,Fx"[i1j]")" * Temper"[ij]" - "max(0,-Fx"[i1j]")" * Temper"[i1j]" + "Fx[i1j]" *
       "TVD_s_in_coeff(Temper[i_1j],Temper[ij],Temper[i1j],Temper[i2j],hx[i_1],hx[i],hx[i1
         ],hx[i2],Fx[i1j])" * (Temper"[i1j]" - Temper"[ij]") +
     -("max(0,Fx"[ij]")" * Temper"[i_1j]" - "max(0,-Fx"[ij]")" * Temper"[ij]" + "Fx[ij]" *
         "TVD_s_in_coeff(Temper[i_2j],Temper[i_1j],Temper[ij],Temper[i1j],hx[i_2],hx[i_1
           ],hx[i],hx[i1],Fx[ij])" *
         (Temper"[ij]" - Temper"[i_1j]"))) * ht];
```

In[86]:=
```
ITdrhovTdy =
  Simplify[("max(0,Fy"[ij1]")" * Temper"[ij]" - "max(0,-Fy"[ij1]")" * Temper"[ij1]" + "Fy[ij1]" *
       "TVD_s_in_coeff(Temper[ij_1],Temper[ij],Temper[ij1],Temper[ij2],hy[j_1],hy[j],hy[j1
         ],hy[j2],Fy[ij1])" * (Temper"[ij1]" - Temper"[ij]") -
     ("max(0,Fy"[ij]")" * Temper"[ij_1]" - "max(0,-Fy"[ij]")" * Temper"[ij]" + "Fy[ij]" *
        "TVD_s_in_coeff(Temper[ij_2],Temper[ij_1],Temper[ij],Temper[ij1],hy[j_2],hy[j_1],
          hy[j],hy[j1],Fy[ij])" *
        (Temper"[ij]" - Temper"[ij_1]"))) * ht];
```

In[87]:=
```
(* Integration of diffusion terms *)
```

In[88]:=
```
(*
DTx"[ij]"=CT1*Γλ"x_i^f"*hy"[j]"/(0.5*(hx"[i_1]"+hx"[i]"));

Γλ"x_i^f" is determined using average harmonic between two values:
   Γλ"x_i^f"=Hi(Γλ"[i_1j]",Γλ"[ij]",hx"[i_1]",hx"[i]");
*)
ITdGldTdx2 =
  Simplify[(DTx"[i1j]" * (Temper"[i1j]" - Temper"[ij]") - DTx"[ij]" * (Temper"[ij]" - Temper"[i_1j]")) * ht];
```

In[89]:= (*

$$DTy_{\text{"[ij]"}}=CT1*\Gamma^{\lambda}{}_{\text{"}y_j^f\text{"}}*\frac{hx_{\text{"[i]"}}}{0.5*(hy_{\text{"[j\_1]"}}+hy_{\text{"[j]"}})};$$

$\Gamma^{\lambda}{}_{\text{"}y_j^f\text{"}}$ is determined using average harmonic between two values:

$$\Gamma^{\lambda}{}_{\text{"}y_j^f\text{"}}=Hi(\Gamma^{\lambda}_{\text{"[ij\_1]"}},\Gamma^{\lambda}_{\text{"[ij]"}},hy_{\text{"[j\_1]"}},hy_{\text{"[j]"}});$$

*)

$$ITdGldTdy2 = Simplify[(DTy_{\text{"[ij1]"}}*(Temper_{\text{"[ij1]"}}-Temper_{\text{"[ij]"}})-DTy_{\text{"[ij]"}}*(Temper_{\text{"[ij]"}}-Temper_{\text{"[ij\_1]"}}))*ht];$$

In[90]:= (* Integrate source term *)

In[91]:=
$$ITdudx2=\left(\frac{u_{\text{"[i1j]"}}-u_{\text{"[ij]"}}}{hx_{\text{"[i]"}}}\right)^2*hx_{\text{"[i]"}}*hy_{\text{"[j]"}};$$

In[92]:=
$$ITdvdy2=\left(\frac{v_{\text{"[ij1]"}}-v_{\text{"[ij]"}}}{hy_{\text{"[j]"}}}\right)^2*hx_{\text{"[i]"}}*hy_{\text{"[j]"}};$$

In[93]:=
$$ITdvdxdudy2 = Simplify\Bigg[\Bigg(\left(\frac{v_{\text{"[i1j]"}}-v_{\text{"[ij]"}}}{\frac{1}{2}*(hx_{\text{"[i]"}}+hx_{\text{"[i1]"}})}\right)^2+\left(\frac{v_{\text{"[ij]"}}-v_{\text{"[i\_1j]"}}}{\frac{1}{2}*(hx_{\text{"[i\_1]"}}+hx_{\text{"[i]"}})}\right)^2+\left(\frac{v_{\text{"[i1j1]"}}-v_{\text{"[ij1]"}}}{\frac{1}{2}*(hx_{\text{"[i]"}}+hx_{\text{"[i1]"}})}\right)^2+$$
$$\left(\frac{v_{\text{"[i\_1j1]"}}-v_{\text{"[ij1]"}}}{\frac{1}{2}*(hx_{\text{"[i\_1]"}}+hx_{\text{"[i]"}})}\right)^2+\left(\frac{u_{\text{"[ij1]"}}-u_{\text{"[ij]"}}}{\frac{1}{2}*(hy_{\text{"[j]"}}+hy_{\text{"[j1]"}})}\right)^2+\left(\frac{u_{\text{"[ij]"}}-u_{\text{"[ij\_1]"}}}{\frac{1}{2}*(hy_{\text{"[j\_1]"}}+hy_{\text{"[j]"}})}\right)^2+$$
$$\left(\frac{u_{\text{"[i1j1]"}}-u_{\text{"[i1j]"}}}{\frac{1}{2}*(hy_{\text{"[j]"}}+hy_{\text{"[j1]"}})}\right)^2+\left(\frac{u_{\text{"[i1j]"}}-u_{\text{"[i1j\_1]"}}}{\frac{1}{2}*(hy_{\text{"[j\_1]"}}+hy_{\text{"[j]"}})}\right)^2\Bigg)*\frac{1}{2}*hx_{\text{"[i]"}}*\frac{1}{2}*hy_{\text{"[j]"}}\Bigg];$$

In[94]:=
$$ITdudxdvdy2=\left(\frac{u_{\text{"[i1j]"}}-u_{\text{"[ij]"}}}{hx_{\text{"[i]"}}}+\frac{v_{\text{"[ij1]"}}-v_{\text{"[ij]"}}}{hy_{\text{"[j]"}}}\right)^2*hx_{\text{"[i]"}}*hy_{\text{"[j]"}};$$

In[95]:=
$$IT\Phi = Simplify\left[\left(2*(ITdudx2+ITdvdy2)+ITdvdxdudy2-\frac{2}{3}*ITdudxdvdy2\right)\right]$$

Out[95]=
$$hx_{[i]}\,hy_{[j]}\Bigg(\frac{(u_{[i1j]}-u_{[i1j\_1]})^2}{(hy_{[j]}+hy_{[j\_1]})^2}+\frac{(u_{[i1j]}-u_{[i1j1]})^2}{(hy_{[j]}+hy_{[j1]})^2}+\frac{(u_{[ij]}-u_{[ij\_1]})^2}{(hy_{[j]}+hy_{[j\_1]})^2}+$$
$$\frac{(u_{[ij]}-u_{[ij1]})^2}{(hy_{[j]}+hy_{[j1]})^2}+\frac{(v_{[i\_1j]}-v_{[ij]})^2}{(hx_{[i]}+hx_{[i\_1]})^2}+\frac{(v_{[i1j]}-v_{[ij]})^2}{(hx_{[i]}+hx_{[i1]})^2}+\frac{(v_{[i\_1j1]}-v_{[ij1]})^2}{(hx_{[i]}+hx_{[i\_1]})^2}+\frac{(v_{[i1j1]}-v_{[ij1]})^2}{(hx_{[i]}+hx_{[i1]})^2}\Bigg)+$$
$$2\left(\frac{hy_{[j]}\,(u_{[i1j]}-u_{[ij]})^2}{hx_{[i]}}+\frac{hx_{[i]}\,(v_{[ij]}-v_{[ij1]})^2}{hy_{[j]}}\right)-\frac{2}{3}\,hx_{[i]}\,hy_{[j]}\left(\frac{u_{[i1j]}-u_{[ij]}}{hx_{[i]}}+\frac{-v_{[ij]}+v_{[ij1]}}{hy_{[j]}}\right)^2$$

In[96]:=
$$ITdudx=(u_{\text{"[i1j]"}}-u_{\text{"[ij]"}})*hy_{\text{"[j]"}};$$

In[97]:=
$$ITdvdy=(v_{\text{"[ij1]"}}-v_{\text{"[ij]"}})*hx_{\text{"[i]"}};$$

In[98]:=
```
(*
The source term is: ST = C_T2.Γ.Φ+C_T3.p(∂_x u+∂_x v)
     where:
       Φ=2((∂_x u)^2+(∂_y v)^2)+(∂_x v+∂_y u)^2-2/3(∂_x u+∂_y v)^2
 *)
ITST = CT2 * Γ * ITΦ + CT2 * p"[ij]" * (ITdudx + ITdvdy)
```

Out[98]=
$$CT2\, p_{[i\,j]}\left(hy_{[j]}\left(u_{[i1j]}-u_{[i\,j]}\right)+hx_{[i]}\left(-v_{[i\,j]}+v_{[i\,j1]}\right)\right)+ CT2\,\Gamma\left(hx_{[i]}\,hy_{[j]}\left(\frac{\left(u_{[i1j]}-u_{[i1j\_1]}\right)^2}{\left(hy_{[j]}+hy_{[j\_1]}\right)^2}+\frac{\left(u_{[i1j]}-u_{[i1j1]}\right)^2}{\left(hy_{[j]}+hy_{[j1]}\right)^2}+\frac{\left(u_{[i\,j]}-u_{[i\,j\_1]}\right)^2}{\left(hy_{[j]}+hy_{[j\_1]}\right)^2}+\frac{\left(u_{[i\,j]}-u_{[i\,j1]}\right)^2}{\left(hy_{[j]}+hy_{[j1]}\right)^2}+\frac{\left(v_{[i\_1j]}-v_{[i\,j]}\right)^2}{\left(hx_{[i]}+hx_{[i\_1]}\right)^2}+\frac{\left(v_{[i1j]}-v_{[i\,j]}\right)^2}{\left(hx_{[i]}+hx_{[i1]}\right)^2}+\frac{\left(v_{[i\_1j1]}-v_{[i\,j1]}\right)^2}{\left(hx_{[i]}+hx_{[i\_1]}\right)^2}+\frac{\left(v_{[i1j1]}-v_{[i\,j1]}\right)^2}{\left(hx_{[i]}+hx_{[i1]}\right)^2}\right)+ 2\left(\frac{hy_{[j]}\left(u_{[i1j]}-u_{[i\,j]}\right)^2}{hx_{[i]}}+\frac{hx_{[i]}\left(v_{[i\,j]}-v_{[i\,j1]}\right)^2}{hy_{[j]}}\right)-\frac{2}{3}\,hx_{[i]}\,hy_{[j]}\left(\frac{u_{[i1j]}-u_{[i\,j]}}{hx_{[i]}}+\frac{-v_{[i\,j]}+v_{[i\,j1]}}{hy_{[j]}}\right)^2\right)$$

In[99]:=
```
(* Derive coefficients for source term *)
```

In[100]:=
```
aST0 = Simplify[-Coefficient[ITST, Temper"[ij]"]]
```

Out[100]= 0

In[101]:=
```
aST1 = Simplify[Coefficient[ITST, Temper"[i_1j]"]]
```

Out[101]= 0

In[102]:=
```
aST2 = Simplify[Coefficient[ITST, Temper"[i1j]"]]
```

Out[102]= 0

In[103]:=
```
aST3 = Simplify[Coefficient[ITST, Temper"[ij_1]"]]
```

Out[103]= 0

In[104]:=
```
aST4 = Simplify[Coefficient[ITST, Temper"[ij1]"]]
```

Out[104]= 0

In[105]:=
```
(* STp = 0 => the coefficients aST0, aST1, aST2,
aST3 and aST4 are not check for simplification. *)
```

In[106]:=
```
(* STp = 0 and all coefficients aST0, aST1, aST2, aST3 and aST4 are 0.
 Therefore STc is equal to the integrated source terms of energy equation. *)
STc = ITST
```

Out[106]=

$$CT2\, p_{[i\,j]} \left(hy_{[j]} \left(u_{[i1\,j]} - u_{[i\,j]}\right) + hx_{[i]} \left(-v_{[i\,j]} + v_{[i\,j1]}\right)\right) + CT2\, \Gamma \left(hx_{[i]}\, hy_{[j]} \left(\frac{\left(u_{[i1\,j]} - u_{[i1\,j\_1]}\right)^2}{\left(hy_{[j]} + hy_{[j\_1]}\right)^2} + \frac{\left(u_{[i1\,j]} - u_{[i1\,j1]}\right)^2}{\left(hy_{[j]} + hy_{[j1]}\right)^2} + \frac{\left(u_{[i\,j]} - u_{[i\,j\_1]}\right)^2}{\left(hy_{[j]} + hy_{[j\_1]}\right)^2} + \frac{\left(u_{[i\,j]} - u_{[i\,j1]}\right)^2}{\left(hy_{[j]} + hy_{[j1]}\right)^2} + \frac{\left(v_{[i\_1\,j]} - v_{[i\,j]}\right)^2}{\left(hx_{[i]} + hx_{[i\_1]}\right)^2} + \frac{\left(v_{[i1\,j]} - v_{[i\,j]}\right)^2}{\left(hx_{[i]} + hx_{[i1]}\right)^2} + \frac{\left(v_{[i\_1\,j1]} - v_{[i\,j1]}\right)^2}{\left(hx_{[i]} + hx_{[i\_1]}\right)^2} + \frac{\left(v_{[i1\,j1]} - v_{[i\,j1]}\right)^2}{\left(hx_{[i]} + hx_{[i1]}\right)^2}\right) + 2 \left(\frac{hy_{[j]} \left(u_{[i1\,j]} - u_{[i\,j]}\right)^2}{hx_{[i]}} + \frac{hx_{[i]} \left(v_{[i\,j]} - v_{[i\,j1]}\right)^2}{hy_{[j]}}\right) - \frac{2}{3}\, hx_{[i]}\, hy_{[j]} \left(\frac{u_{[i1\,j]} - u_{[i\,j]}}{hx_{[i]}} + \frac{-v_{[i\,j]} + v_{[i\,j1]}}{hy_{[j]}}\right)^2\right)$$

In[107]:=
```
(**)
```

In[108]:=
```
(* All terms are moved to the left hand
 side to derive the numerical coefficients. *)
TExpresion = Simplify[(ITdrhoTdt + ITdrhouTdx + ITdrhovTdy - (ITdGldTdx2 + ITdGldTdy2)) - STc]
```

Out[108]=

$$-ht\left(DTx_{[ij]}\left(Temper_{[i\_1j]}-Temper_{[ij]}\right)+DTx_{[i1j]}\left(Temper_{[i1j]}-Temper_{[ij]}\right)\right)+ht\Big(\big(-\max(0,Fx_{[ij]})+Fx[ij]\ \text{TVD\_s\_in\_coeff}(Temper[i\_2j],Temper[i\_1j],Temper[ij],Temper[i1j],hx[i\_2],hx[i\_1],hx[i],hx[i1],Fx[ij])\big)Temper_{[i\_1j]}+$$
$$\big(-\max(0,-Fx_{[i1j]})+Fx[i1j]\ \text{TVD\_s\_in\_coeff}(Temper[i\_1j],Temper[ij],Temper[i1j],Temper[i2j],hx[i\_1],hx[i],hx[i1],hx[i2],Fx[i1j])\big)Temper_{[i1j]}+$$
$$\big(\max(0,Fx_{[i1j]})+\max(0,-Fx_{[ij]})-Fx[i1j]\ \text{TVD\_s\_in\_coeff}(Temper[i\_1j],Temper[ij],Temper[i1j],Temper[i2j],hx[i\_1],hx[i],hx[i1],hx[i2],Fx[i1j])-Fx[ij]\ \text{TVD\_s\_in\_coeff}(Temper[i\_2j],Temper[i\_1j],Temper[ij],Temper[i1j],hx[i\_2],hx[i\_1],hx[i],hx[i1],Fx[ij])\big)Temper_{[ij]}\Big)+$$
$$ht\Big(\big(\max(0,-Fy_{[ij]})+\max(0,Fy_{[ij1]})-Fy[ij1]\ \text{TVD\_s\_in\_coeff}(Temper[ij\_1],Temper[ij],Temper[ij1],Temper[ij2],hy[j\_1],hy[j],hy[j1],hy[j2],Fy[ij1])-Fy[ij]\ \text{TVD\_s\_in\_coeff}(Temper[ij\_2],Temper[ij\_1],Temper[ij],Temper[ij1],hy[j\_2],hy[j\_1],hy[j],hy[j1],Fy[ij])\big)Temper_{[ij]}+\big(-\max(0,Fy_{[ij]})+Fy[ij]\ \text{TVD\_s\_in\_coeff}(Temper[ij\_2],Temper[ij\_1],Temper[ij],Temper[ij1],hy[j\_2],hy[j\_1],hy[j],hy[j1],Fy[ij])\big)Temper_{[ij\_1]}+$$
$$\big(-\max(0,-Fy_{[ij1]})+Fy[ij1]\ \text{TVD\_s\_in\_coeff}(Temper[ij\_1],Temper[ij],Temper[ij1],Temper[ij2],hy[j\_1],hy[j],hy[j1],hy[j2],Fy[ij1])\big)Temper_{[ij1]}\Big)-$$
$$ht\left(DTy_{[ij]}\left(-Temper_{[ij]}+Temper_{[ij\_1]}\right)+DTy_{[ij1]}\left(-Temper_{[ij]}+Temper_{[ij1]}\right)\right)+$$
$$hx_{[i]}\,hy_{[j]}\left(rho_{[ij]}\,Temper_{[ij]}-rhopr_{[ij]}\,Temperpr_{[ij]}\right)-CT2\,p_{[ij]}\left(hy_{[j]}\left(u_{[i1j]}-u_{[ij]}\right)+hx_{[i]}\left(-v_{[ij]}+v_{[ij1]}\right)\right)-$$
$$CT2\,\Gamma\left(hx_{[i]}\,hy_{[j]}\left(\frac{\left(u_{[i1j]}-u_{[i1j\_1]}\right)^2}{\left(hy_{[j]}+hy_{[j\_1]}\right)^2}+\frac{\left(u_{[i1j]}-u_{[i1j1]}\right)^2}{\left(hy_{[j]}+hy_{[j1]}\right)^2}+\frac{\left(u_{[ij]}-u_{[ij\_1]}\right)^2}{\left(hy_{[j]}+hy_{[j\_1]}\right)^2}+\frac{\left(u_{[ij]}-u_{[ij1]}\right)^2}{\left(hy_{[j]}+hy_{[j1]}\right)^2}+\right.\right.$$
$$\left.\frac{\left(v_{[i\_1j]}-v_{[ij]}\right)^2}{\left(hx_{[i]}+hx_{[i\_1]}\right)^2}+\frac{\left(v_{[i1j]}-v_{[ij]}\right)^2}{\left(hx_{[i]}+hx_{[i1]}\right)^2}+\frac{\left(v_{[i\_1j1]}-v_{[ij1]}\right)^2}{\left(hx_{[i]}+hx_{[i\_1]}\right)^2}+\frac{\left(v_{[i1j1]}-v_{[ij1]}\right)^2}{\left(hx_{[i]}+hx_{[i1]}\right)^2}\right)+$$
$$\left.2\left(\frac{hy_{[j]}\left(u_{[i1j]}-u_{[ij]}\right)^2}{hx_{[i]}}+\frac{hx_{[i]}\left(v_{[ij]}-v_{[ij1]}\right)^2}{hy_{[j]}}\right)-\frac{2}{3}\,hx_{[i]}\,hy_{[j]}\left(\frac{u_{[i1j]}-u_{[ij]}}{hx_{[i]}}+\frac{-v_{[ij]}+v_{[ij1]}}{hy_{[j]}}\right)^2\right)$$

In[109]:=
```
(* Derive numerical coefficients *)
aT0 = Simplify[Coefficient[TExpresion, Temper"[ij]"]]
```

Out[109]=
```
max(0, Fx[i1j]) ht + max(0, -Fx[ij]) ht + max(0, -Fy[ij]) ht + max(0, Fy[ij1]) ht - Fx[i1j]
  TVD_s_in_coeff(Temper[i_1j], Temper[ij], Temper[i1j], Temper[i2j], hx[i_1], hx[i], hx[i1], hx[i2],
    Fx[i1j]) ht - Fx[ij]
  TVD_s_in_coeff(Temper[i_2j], Temper[i_1j], Temper[ij], Temper[i1j], hx[i_2], hx[i_1], hx[i], hx[i1]
    , Fx[ij]) ht - Fy[ij1]
  TVD_s_in_coeff(Temper[ij_1], Temper[ij], Temper[ij1], Temper[ij2], hy[j_1], hy[j], hy[j1], hy[j2],
    Fy[ij1]) ht - Fy[ij]
  TVD_s_in_coeff(Temper[ij_2], Temper[ij_1], Temper[ij], Temper[ij1], hy[j_2], hy[j_1], hy[j], hy[j1]
    , Fy[ij]) ht + ht DTx[i1j] + ht DTx[ij] + ht DTy[ij] + ht DTy[ij1] + hx[i] hy[j] rho[ij]
```

In[110]:=
```
aT1 = Simplify[-Coefficient[TExpresion, Temper"[i_1j]"]]
```

Out[110]=
```
ht (max(0, Fx[ij]) - Fx[ij]
    TVD_s_in_coeff(Temper[i_2j], Temper[i_1j], Temper[ij], Temper[i1j], hx[i_2], hx[i_1], hx[i], hx[
      i1], Fx[ij]) + DTx[ij])
```

In[111]:=
```
aT2 = Simplify[-Coefficient[TExpresion, Temper"[i1j]"]]
```

Out[111]=
```
ht (max(0, -Fx[i1j]) - Fx[i1j]
    TVD_s_in_coeff(Temper[i_1j], Temper[ij], Temper[i1j], Temper[i2j], hx[i_1], hx[i], hx[i1], hx[i2]
      , Fx[i1j]) + DTx[i1j])
```

In[112]:=
```
aT3 = Simplify[-Coefficient[TExpresion, Temper"[ij_1]"]]
```

Out[112]=
```
ht (max(0, Fy[ij]) - Fy[ij]
    TVD_s_in_coeff(Temper[ij_2], Temper[ij_1], Temper[ij], Temper[ij1], hy[j_2], hy[j_1], hy[j], hy[
      j1], Fy[ij]) + DTy[ij])
```

In[113]:=
```
aT4 = Simplify[-Coefficient[TExpresion, Temper"[ij1]"]]
```

Out[113]=
```
ht (max(0, -Fy[ij1]) - Fy[ij1]
    TVD_s_in_coeff(Temper[ij_1], Temper[ij], Temper[ij1], Temper[ij2], hy[j_1], hy[j], hy[j1], hy[j2]
      , Fy[ij1]) + DTy[ij1])
```

In[114]:=
```
bT = Simplify[-(TExpresion - (aT0 * Temper"[ij]" -
      (aT1 * Temper"[i_1j]" + aT2 * Temper"[i1j]" + aT3 * Temper"[ij_1]" + aT4 * Temper"[ij1]" + STc)))]
```

Out[114]=
```
hx[i] hy[j] rhopr[ij] Temperpr[ij]
```

In[115]:=
```
(* Check the derived numerical coefficients - the result has to be zero: *)
Simplify[TExpresion - (aT0 * Temper"[ij]" -
    (aT1 * Temper"[i_1j]" + aT2 * Temper"[i1j]" + aT3 * Temper"[ij_1]" + aT4 * Temper"[ij1]" + bT + STc))]
```

Out[115]=
```
0
```