In[1]:= (*

% Developed and updated by Assoc.Prof.Dr.Eng.Kiril Shterev.

% April, 4th,2022.

%

%

% Please cite my papers if you find this information useful:

%

% K.Shterev and S.Stefanov,Pressure based finite volume method
% for calculation of compressible viscous gas flows,Journal of
% Computational Physics 229 (2010) pp.461-480,doi:10.1016/j.jcp.2009.09.042

%

% K.S.Shterev and S.K.Stefanov,A Parallelization of Finite Volume Method
% for Calculation of Gas Microflows by Domain Decomposition Methods,7th
% Internnational Conference-Large-ScaleScientific Computations,Sozopol,
Bulgaria,June 04-08,2009. Lecture Notes in Computer Science Volume 5910,
% 2010,DOI:10.1007/978-3-642-12535-5,SJR 0.295.

%

% Kiril S.Shterev,GPU implementation of algorithm SIMPLE-TS for calculation
% of unsteady,viscous,compressible and heat-conductive gas flows,
% URL:https://arxiv.org/abs/1802.04243,2018.

%

Derive numerical equations of partial differencial equations of viscous,
compressible, heat conductive gas for 2D case,
according SIMPLE-TS published in Journal of Computational Physics,
2010, doi:10.1016/j.jcp.2009.09.042 *)

(* The system of PDE equations is:;

$$\partial_t(\rho.u)+\partial_x(\rho.u.u)+\partial_y(\rho.v.u)=-A\partial_x p+B(\partial_x(\Gamma\partial_x u)+\partial_y(\Gamma\partial_y u))+\rho.g_x+B\left(\partial_x(\Gamma\partial_x u)+\partial_y(\Gamma\partial_x v)-\frac{2}{3}\partial_x(\Gamma(\partial_x u+\partial_y v))\right)$$

;

$$\partial_t(\rho.v)+\partial_x(\rho.u.v)+\partial_y(\rho.v.v)=-A\partial_y p+B(\partial_x(\Gamma\partial_x v)+\partial_y(\Gamma\partial_y v))+\rho.g_y+B\left(\partial_y(\Gamma\partial_y v)+\partial_x(\Gamma\partial_y u)-\frac{2}{3}\partial_y(\Gamma(\partial_x u+\partial_y v))\right)$$

;

$$\partial_t\rho+\partial_x(\rho.u)+\partial_y(\rho.v)=0$$

;

$$\partial_t(\rho.T)+\partial_x(\rho.u.T)+\partial_y(\rho.v.T)=C_{T1}(\partial_x(\Gamma_\lambda\partial_x T)+\partial_y(\Gamma_\lambda\partial_y T))+C_{T2}.\Gamma.\Phi+C_{T3}.p(\partial_x u+\partial_x v)$$

where:

$$\Phi=2\left((\partial_x u)^2+(\partial_y v)^2\right)+(\partial_x v+\partial_y u)^2-\frac{2}{3}(\partial_x u+\partial_y v)^2$$

*)

In[2]:= (* Integration of equation for u *)

In[3]:=
```
(* Integration of unsteady term *)
Iudρudt = hy"[j]"/(2 * ht) ((rho"[i_1j]" * hx"[i_1]" + rho"[ij]" * hx"[i]") * u"[ij]" -
    (rhopr"[i_1j]" * hx"[i_1]" + rhopr"[ij]" * hx"[i]") * upr"[ij]");
```

In[4]:=
```
(* Integration of convective terms *)
```

In[5]:=
```
(*  F1x"[ij]" is defined at a point (x_v"[i]", y_v"[j]"),
where field variables are defined;
F1x"[ij]" = hy"[j]"*rho"[ij]"*1/2*(u"[i1j]"+u"[ij]") - in new definition,
it is used that rho is defined at a Control Volume Surface x_v"[i]";
F1x"[ij]" = 1/2*(Fx"[i1j]"+Fx"[ij]") - old definition  *)
Iudρuudx = Simplify["max(0,F1x"[ij]")" * u"[ij]" - "max(0,-F1x"[ij]")" * u"[i1j]" -
    ("max(0,F1x"[i_1j]")" * u"[i_1j]" - "max(0,-F1x"[i_1j]")" * u"[ij]")];
```

In[6]:=
```
Iudρvudy = Simplify[1/2 * ("max(0,Fy"[i_1j1]")" * u"[ij]" -
    "max(0,-Fy"[i_1j1]")" * u"[ij1]" + "max(0,Fy"[ij1]")" * u"[ij]" - "max(0,-Fy"[ij1]")" * u"[ij1]"
    - ("max(0,Fy"[i_1j]")" * u"[ij_1]" - "max(0,-Fy"[i_1j]")" * u"[ij]" +
      "max(0,Fy"[ij]")" * u"[ij_1]" - "max(0,-Fy"[ij]")" * u"[ij]"))]
```

Out[6]=

$$\frac{1}{2}\left(\left(\max(0,-Fy_{[i\_1j]})+\max(0,Fy_{[i\_1j1]})+\max(0,-Fy_{[ij]})+\max(0,Fy_{[ij1]})\right)u_{[ij]}-\left(\max(0,Fy_{[i\_1j]})+\max(0,Fy_{[ij]})\right)u_{[ij\_1]}-\left(\max(0,-Fy_{[i\_1j1]})+\max(0,-Fy_{[ij1]})\right)u_{[ij1]}\right)$$

In[7]:=
```
(* Integration of diffusion terms *)
```

In[8]:=
```
(*
Dux"[i1j]"=B*Γ"[ij]"*hy"[j]"/hx"[i]";
Dux"[ij]"=B*Γ"[i_1j]"*hy"[j]"/hx"[i_1]";
*)
IudΓdudx2 = Dux"[i1j]" * (u"[i1j]" - u"[ij]") - Dux"[ij]" * (u"[ij]" - u"[i_1j]");
(* Interpolation of Γ in middle point is:
   Γyf"[ij]"=Hi(Γ"[ij_1]",Γ"[ij]",hy"[j_1]",hy"[j]")
*)
(*
Duy"[ij]"=B*(hx"[i]"*Γyf"[i_1j]"+hx"[i]"*Γyf"[ij]")*1/(hy"[j_1]"+hy"[j]");
Duy"[ij1]"=B*(hx"[i]"*Γyf"[i_1j1]"+hx"[i]"*Γyf"[ij1]")*1/(hy"[j]"+hy"[j1]");
*)
IudΓdudy2 = Duy"[ij1]" * (u"[ij1]" - u"[ij]") - Duy"[ij]" * (u"[ij]" - u"[ij_1]");
```

In[10]:=
```
(* Integration of pressure term *)
Iudpdx = -A * (p"[ij]" - p"[i_1j]") * hy"[j]";
```

In[11]:=
```
(* Integration of source term *)
IudΓdvdydx = B * ((Γyf"[i_1j1]" * hx"[i_1]" + Γyf"[ij1]" * hx"[i]") / (hx"[i_1]" + hx"[i]") * (v"[ij1]" - v"[i_1j1]") -
    (Γyf"[i_1j]" * hx"[i_1]" + Γyf"[ij]" * hx"[i]") / (hx"[i_1]" + hx"[i]") * (v"[ij]" - v"[i_1j]"));
IudΓdvdxdy = B * (Γ"[ij]" * (v"[ij1]" - v"[ij]") - Γ"[i_1j]" * (v"[i_1j1]" - v"[i_1j]"));
```

In[13]:=
```
(* The source term is: Su = B(∂x(Γ∂xu)+∂y(Γ∂xv)-2/3∂x(Γ(∂xu+∂yv))) *)
IuSu = IudΓdudx2 + IudΓdvdydx - 2/3 * (IudΓdudx2 + IudΓdvdxdy)
```

Out[13]=

$$Dux_{[i1j]}\left(u_{[i1j]}-u_{[ij]}\right)-Dux_{[ij]}\left(-u_{[i\_1j]}+u_{[ij]}\right)-\frac{2}{3}\left(Dux_{[i1j]}\left(u_{[i1j]}-u_{[ij]}\right)-Dux_{[ij]}\left(-u_{[i\_1j]}+u_{[ij]}\right)+B\left(-\left(\left(-v_{[i\_1j]}+v_{[i\_1j1]}\right)\Gamma_{[i\_1j]}\right)+\left(-v_{[ij]}+v_{[ij1]}\right)\Gamma_{[ij]}\right)\right)+B\left(-\frac{\left(-v_{[i\_1j]}+v_{[ij]}\right)\left(hx_{[i\_1]}\,\Gamma yf_{[i\_1j]}+hx_{[i]}\,\Gamma yf_{[ij]}\right)}{hx_{[i]}+hx_{[i\_1]}}+\frac{\left(-v_{[i\_1j1]}+v_{[ij1]}\right)\left(hx_{[i\_1]}\,\Gamma yf_{[i\_1j1]}+hx_{[i]}\,\Gamma yf_{[ij1]}\right)}{hx_{[i]}+hx_{[i\_1]}}\right)$$

In[14]:=
```
(* Derive the coefficients for source term *)
```

In[15]:=
```
aSu0 = Simplify[-Coefficient[IuSu, u"[ij]"]]
```

Out[15]=

$$\frac{1}{3}\left(Dux_{[i1j]}+Dux_{[ij]}\right)$$

In[16]:=
```
aSu1 = Simplify[Coefficient[IuSu, u"[i_1j]"]]
```

Out[16]=

$$\frac{Dux_{[ij]}}{3}$$

In[17]:=
```
aSu2 = Simplify[Coefficient[IuSu, u"[i1j]"]]
```

Out[17]=

$$\frac{Dux_{[i1j]}}{3}$$

In[18]:=
```
aSu3 = Simplify[Coefficient[IuSu, u"[ij_1]"]]
```

Out[18]= 0

In[19]:=
```
aSu4 = Simplify[Coefficient[IuSu, u"[ij1]"]]
```

Out[19]= 0

In[20]:=
```
Suc =
  Simplify[-(IuSu - (aSu0 * u"[ij]" - (aSu1 * u"[i_1j]" + aSu2 * u"[i1j]" + aSu3 * u"[ij_1]" + aSu4 * u"[ij1]")))]
```

Out[20]=

$$\frac{1}{3\,(hx_{[i]} + hx_{[i\_1]})} \Big(-2\,Dux_{[ij]}\,(hx_{[i]} + hx_{[i\_1]})\,(u_{[i\_1j]} - u_{[ij]}) - 2\,Dux_{[i1j]}\,(hx_{[i]} + hx_{[i\_1]})\,(u_{[i1j]} - u_{[ij]}) + B\,hx_{[i\_1]}\,\big(-2\,v_{[ij]}\,\Gamma_{[ij]} + 2\,v_{[ij1]}\,\Gamma_{[ij]} + v_{[i\_1j]}\,(2\,\Gamma_{[i\_1j]} - 3\,\Gamma yf_{[i\_1j]}) + 3\,v_{[ij]}\,\Gamma yf_{[i\_1j]} - 3\,v_{[ij1]}\,\Gamma yf_{[i\_1j1]} + v_{[i\_1j1]}\,(-2\,\Gamma_{[i\_1j]} + 3\,\Gamma yf_{[i\_1j1]})\big) + B\,hx_{[i]}\,\big(-2\,v_{[ij]}\,\Gamma_{[ij]} + 2\,v_{[ij1]}\,\Gamma_{[ij]} + v_{[i\_1j]}\,(2\,\Gamma_{[i\_1j]} - 3\,\Gamma yf_{[ij]}) + 3\,v_{[ij]}\,\Gamma yf_{[ij]} - 3\,v_{[ij1]}\,\Gamma yf_{[ij1]} + v_{[i\_1j1]}\,(-2\,\Gamma_{[i\_1j]} + 3\,\Gamma yf_{[ij1]})\big)\Big)$$

In[21]:=
```
(* Check derived numerical coefficients - the result have to be zero: *)
Simplify[IuSu - (aSu0 * u"[ij]" - (aSu1 * u"[i_1j]" + aSu2 * u"[i1j]" + aSu3 * u"[ij_1]" + aSu4 * u"[ij1]" + Suc))]
```

Out[21]= 0

In[22]:=
```
(**)
```

In[23]:=
```
(* All tetms are moved to the left hand side to derive numerical coefficients. *)
uExpresion =
  FullSimplify[(IudρudT + Iudρuudx + Iudρvudy + Iudpdx - (IudΓdudx2 + IudΓdudy2)) - IuSu]
```

Out[23]=

$$\frac{1}{6}\Bigg(6\,A\,hy_{[j]}\,(p_{[i\_1j]} - p_{[ij]}) - 6\max(0, F1x_{[i\_1j]})\,u_{[i\_1j]} - 6\max(0, -F1x_{[ij]})\,u_{[i1j]} + 6\,(\max(0, -F1x_{[i\_1j]}) + \max(0, F1x_{[ij]}))\,u_{[ij]} + 3\,(\max(0, -Fy_{[i\_1j]}) + \max(0, Fy_{[i\_1j1]}) + \max(0, -Fy_{[ij]}) + \max(0, Fy_{[ij1]}))\,u_{[ij]} + \frac{3\,hy_{[j]}\,(hx_{[i\_1]}\,rho_{[i\_1j]} + hx_{[i]}\,rho_{[ij]})\,u_{[ij]}}{ht} + 8\,Dux_{[ij]}\,(-u_{[i\_1j]} + u_{[ij]}) + 8\,Dux_{[i1j]}\,(-u_{[i1j]} + u_{[ij]}) + 6\,Duy_{[ij]}\,(u_{[ij]} - u_{[ij\_1]}) - 3\,(\max(0, Fy_{[i\_1j]}) + \max(0, Fy_{[ij]}))\,u_{[ij\_1]} + 6\,Duy_{[ij1]}\,(u_{[ij]} - u_{[ij1]}) - 3\,(\max(0, -Fy_{[i\_1j1]}) + \max(0, -Fy_{[ij1]}))\,u_{[ij1]} - \frac{3\,hy_{[j]}\,(hx_{[i\_1]}\,rhopr_{[i\_1j]} + hx_{[i]}\,rhopr_{[ij]})\,upr_{[ij]}}{ht} + 4\,B\,(v_{[i\_1j]} - v_{[i\_1j1]})\,\Gamma_{[i\_1j]} + 4\,B\,(-v_{[ij]} + v_{[ij1]})\,\Gamma_{[ij]} + \frac{6\,B\,(-v_{[i\_1j]} + v_{[ij]})\,(hx_{[i\_1]}\,\Gamma yf_{[i\_1j]} + hx_{[i]}\,\Gamma yf_{[ij]})}{hx_{[i]} + hx_{[i\_1]}} + \frac{6\,B\,(v_{[i\_1j1]} - v_{[ij1]})\,(hx_{[i\_1]}\,\Gamma yf_{[i\_1j1]} + hx_{[i]}\,\Gamma yf_{[ij1]})}{hx_{[i]} + hx_{[i\_1]}}\Bigg)$$

In[24]:=
```
(* Derive numerical coefficients *)
au0 = Simplify[Coefficient[uExpresion, u"[ij]"]]
```

Out[24]=
$$\frac{1}{6}\left(8\,Dux_{[i1j]} + 8\,Dux_{[ij]} + 3\left(2\max(0,-F1x_{[i\_1j]}) + 2\max(0,F1x_{[ij]}) + \max(0,-Fy_{[i\_1j]}) + \max(0,Fy_{[i\_1j1]}) + \max(0,-Fy_{[ij]}) + \max(0,Fy_{[ij1]}) + 2\,Duy_{[ij]} + 2\,Duy_{[ij1]} + \frac{hx_{[i\_1]}\,hy_{[j]}\,rho_{[i\_1j]}}{ht} + \frac{hx_{[i]}\,hy_{[j]}\,rho_{[ij]}}{ht}\right)\right)$$

In[25]:=
```
au1 = Simplify[-Coefficient[uExpresion, u"[i_1j]"]]
```

Out[25]=
$$\max(0,F1x_{[i\_1j]}) + \frac{4\,Dux_{[ij]}}{3}$$

In[26]:=
```
au2 = Simplify[-Coefficient[uExpresion, u"[i1j]"]]
```

Out[26]=
$$\max(0,-F1x_{[ij]}) + \frac{4\,Dux_{[i1j]}}{3}$$

In[27]:=
```
au3 = Simplify[-Coefficient[uExpresion, u"[ij_1]"]]
```

Out[27]=
$$\frac{1}{2}\left(\max(0,Fy_{[i\_1j]}) + \max(0,Fy_{[ij]}) + 2\,Duy_{[ij]}\right)$$

In[28]:=
```
au4 = Simplify[-Coefficient[uExpresion, u"[ij1]"]]
```

Out[28]=
$$\frac{1}{2}\left(\max(0,-Fy_{[i\_1j1]}) + \max(0,-Fy_{[ij1]}) + 2\,Duy_{[ij1]}\right)$$

In[29]:=
```
bu =
 Simplify[-(uExpresion - (au0 * u"[ij]" - (au1 * u"[i_1j]" + au2 * u"[i1j]" + au3 * u"[ij_1]" + au4 * u"[ij1]")))]
```

Out[29]=
$$\frac{1}{6\,ht\,(hx_{[i]} + hx_{[i\_1]})}\Big(3\,hx_{[i]}^2\,hy_{[j]}\,rhopr_{[ij]}\,upr_{[ij]} + hx_{[i\_1]}\big(hy_{[j]}\left(-6\,A\,ht\,p_{[i\_1j]} + 6\,A\,ht\,p_{[ij]} + 3\,hx_{[i\_1]}\,rhopr_{[i\_1j]}\,upr_{[ij]}\right) + 2\,B\,ht\left(2\,v_{[ij]}\,\Gamma_{[ij]} - 2\,v_{[ij1]}\,\Gamma_{[ij]} - 3\,v_{[ij]}\,\Gamma yf_{[i\_1j]} + v_{[i\_1j]}\left(-2\,\Gamma_{[i\_1j]} + 3\,\Gamma yf_{[i\_1j]}\right) + v_{[i\_1j1]}\left(2\,\Gamma_{[i\_1j]} - 3\,\Gamma yf_{[i\_1j1]}\right) + 3\,v_{[ij1]}\,\Gamma yf_{[i\_1j1]}\right)\big) + hx_{[i]}\big(hy_{[j]}\left(-6\,A\,ht\,p_{[i\_1j]} + 6\,A\,ht\,p_{[ij]} + 3\,hx_{[i\_1]}\left(rhopr_{[i\_1j]} + rhopr_{[ij]}\right)upr_{[ij]}\right) + 2\,B\,ht\left(2\,v_{[ij]}\,\Gamma_{[ij]} - 2\,v_{[ij1]}\,\Gamma_{[ij]} - 3\,v_{[ij]}\,\Gamma yf_{[ij]} + v_{[i\_1j]}\left(-2\,\Gamma_{[i\_1j]} + 3\,\Gamma yf_{[ij]}\right) + v_{[i\_1j1]}\left(2\,\Gamma_{[i\_1j]} - 3\,\Gamma yf_{[ij1]}\right) + 3\,v_{[ij1]}\,\Gamma yf_{[ij1]}\right)\big)\Big)$$

In[30]:=
```
(* Check the derived numerical coefficients - the result have to be zero: *)
Simplify[uExpresion - (au0 * u"[ij]" - (au1 * u"[i_1j]" + au2 * u"[i1j]" + au3 * u"[ij_1]" + au4 * u"[ij1]" + bu))]
```

Out[30]= 0

```
In[31]:= (**)

In[32]:= (* Integration of the equation for v *)

In[33]:= (* Integration of unsteady term *)
         Ivdρvdt = hx"[i]"/(2 * ht) ((rho"[ij_1]" * hy"[j_1]" + rho"[ij]" * hy"[j]") * v"[ij]" -
             (rhopr"[ij_1]" * hy"[j_1]" + rhopr"[ij]" * hy"[j]") * vpr"[ij]");

In[34]:= (* Integration of convective terms *)

In[35]:= Ivdρuvdx = Simplify[1/2 ("max(0,Fx"[i1j]")" * v"[ij]" -
               "max(0,-Fx"[i1j]")" * v"[i1j]" + "max(0,Fx"[i1j_1]")" * v"[ij]" - "max(0,-Fx"[i1j_1]")" * v"[i1j]"
               - ("max(0,Fx"[ij]")" * v"[i_1j]" - "max(0,-Fx"[ij]")" * v"[ij]" +
                 "max(0,Fx"[ij_1]")" * v"[i_1j]" - "max(0,-Fx"[ij_1]")" * v"[ij]"))];

In[36]:= (*  F1y"[ij]" is defined in point (x_v"[i]", y_v"[j]"), where field variables are defined;
         F1y"[ij]" = hx"[i]"*rho"[ij]"*1/2*(v"[ij1]"+v"[ij]") - new definition,
         it is used that rho is defined on Control Surface y_v"[j]";
         F1y"[ij]" = 1/2*(Fy"[ij1]"+Fy"[ij]") - old definition  *)
         Ivdρvvdy = Simplify["max(0,F1y"[ij]")" * v"[ij]" -
             "max(0,-F1y"[ij]")" * v"[ij1]" - "max(0,F1y"[ij_1]")" * v"[ij_1]" + "max(0,-F1y"[ij_1]")" * v"[ij]"];

In[37]:= (* Integration of diffusion terms *)

In[38]:= (* Interpolation of Γ in middle point is:
            Γxf"[ij]"=Hi(Γ"[i_1j]",Γ"[ij]",hx"[i_1]",hx"[i]")
         *)
         (*
         Dvx"[ij]"=B*(hy"[j_1]"*Γxf"[ij_1]"+hy"[j]"*Γxf"[ij]")*1/(hx"[i_1]"+hx"[i]");
         Dvx"[i1j]"=B*(hy"[j_1]"*Γxf"[i1j_1]"+hy"[j]"*Γxf"[i1j]")*1/(hx"[i]"+hx"[i1]");
         *)
         IvdΓdvdx2 = Dvx"[i1j]" * (v"[i1j]" - v"[ij]") - Dvx"[ij]" * (v"[ij]" - v"[i_1j]");
```

In[39]:= (*

$Dvy_{"[ij1]"} = B*\Gamma_{"[ij]"}*\frac{hx_{"[i]"}}{hy_{"[j]"}};$

$Dvy_{"[ij]"} = B*\Gamma_{"[ij\_1]"}*\frac{hx_{"[i]"}}{hy_{"[j\_1]"}};$

*)

$Ivd\Gamma dvdy2 = Dvy_{"[ij1]"}*(v_{"[ij1]"} - v_{"[ij]"}) - Dvy_{"[ij]"}*(v_{"[ij]"} - v_{"[ij\_1]"});$

In[40]:= (* Integration of mass forces term (the gravity term) *)

$Iv\rho gy = (rho_{"[ij\_1]"} * hy_{"[j\_1]"} + rho_{"[ij]"} * hy_{"[j]"}) * \frac{gy * hx_{"[i]"}}{2};$

In[41]:= (* Integration of pressure term *)

$Ivdpdy = -A*(p_{"[ij]"} - p_{"[ij\_1]"}) * hx_{"[i]"};$

In[42]:= (* Integration of source term *)

$Ivd\Gamma dudxdy = B*((\Gamma xf_{"[i1j\_1]"} * hy_{"[j\_1]"} + \Gamma xf_{"[i1j]"} * hy_{"[j]"}) / (hy_{"[j\_1]"} + hy_{"[j]"}) * (u_{"[i1j]"} - u_{"[i1j\_1]"}) - (\Gamma xf_{"[ij\_1]"} * hy_{"[j\_1]"} + \Gamma xf_{"[ij]"} * hy_{"[j]"}) / (hy_{"[j\_1]"} + hy_{"[j]"}) * (u_{"[ij]"} - u_{"[ij\_1]"}));$

$Ivd\Gamma dudydx = B*(\Gamma_{"[ij]"} * (u_{"[i1j]"} - u_{"[ij]"}) - \Gamma_{"[ij\_1]"} * (u_{"[i1j\_1]"} - u_{"[ij\_1]"}));$

In[44]:= (* The source term is: $Sv = B(\partial_y(\Gamma\partial_y v) + \partial_x(\Gamma\partial_y u) - \frac{2}{3}\partial_y(\Gamma(\partial_x u + \partial_y v)))$ *)

$IvSv = Ivd\Gamma dvdy2 + Ivd\Gamma dudxdy - \frac{2}{3} * (Ivd\Gamma dudydx + Ivd\Gamma dvdy2)$

Out[44]= $-Dvy_{[ij]}(v_{[ij]} - v_{[ij\_1]}) + Dvy_{[ij1]}(-v_{[ij]} + v_{[ij1]}) - \frac{2}{3}(-Dvy_{[ij]}(v_{[ij]} - v_{[ij\_1]}) + Dvy_{[ij1]}(-v_{[ij]} + v_{[ij1]}) + B((u_{[i1j]} - u_{[ij]})\Gamma_{[ij]} - (u_{[i1j\_1]} - u_{[ij\_1]})\Gamma_{[ij\_1]})) + B\left(\frac{(u_{[i1j]} - u_{[i1j\_1]})(hy_{[j]}\Gamma xf_{[i1j]} + hy_{[j\_1]}\Gamma xf_{[i1j\_1]})}{hy_{[j]} + hy_{[j\_1]}} - \frac{(u_{[ij]} - u_{[ij\_1]})(hy_{[j]}\Gamma xf_{[ij]} + hy_{[j\_1]}\Gamma xf_{[ij\_1]})}{hy_{[j]} + hy_{[j\_1]}}\right)$

In[45]:= (* Derive numerical coefficients for source term *)

In[46]:= aSv0 = Simplify[-Coefficient[IvSv, $v_{"[ij]"}$]]

Out[46]= $\frac{1}{3}(Dvy_{[ij]} + Dvy_{[ij1]})$

In[47]:= aSv1 = Simplify[Coefficient[IvSv, $v_{"[i\_1j]"}$]]

Out[47]= 0

In[48]:= aSv2 = Simplify[Coefficient[IvSv, $v_{"[i1j]"}$]]

Out[48]= 0

In[49]:= aSv3 = Simplify[Coefficient[IvSv, $v_{"[ij\_1]"}$]]

Out[49]= $\frac{Dvy_{[ij]}}{3}$

In[50]:=
```
aSv4 = Simplify[Coefficient[IvSv, v"[ij1]"]]
```

Out[50]=
$$\frac{Dvy_{[ij1]}}{3}$$

In[51]:=
```
Svc =
  Simplify[-(IvSv - (aSv0 * v"[ij]" - (aSv1 * v"[i_1j]" + aSv2 * v"[i1j]" + aSv3 * v"[ij_1]" + aSv4 * v"[ij1]")))]
```

Out[51]=
$$\frac{1}{3\,(hy_{[j]} + hy_{[j\_1]})}$$
$$(2\,Dvy_{[ij]}\,(hy_{[j]} + hy_{[j\_1]})\,(v_{[ij]} - v_{[ij\_1]}) + 2\,Dvy_{[ij1]}\,(hy_{[j]} + hy_{[j\_1]})\,(v_{[ij]} - v_{[ij1]}) + B\,hy_{[j]}\,(-2\,u_{[i1j\_1]}\,\Gamma_{[ij\_1]} +$$
$$2\,u_{[ij\_1]}\,\Gamma_{[ij\_1]} + u_{[i1j]}\,(2\,\Gamma_{[ij]} - 3\,\Gamma xf_{[i1j]}) + 3\,u_{[i1j\_1]}\,\Gamma xf_{[i1j]} - 3\,u_{[ij\_1]}\,\Gamma xf_{[ij]} + u_{[ij]}\,(-2\,\Gamma_{[ij]} + 3\,\Gamma xf_{[ij]})) +$$
$$B\,hy_{[j\_1]}\,(-2\,u_{[i1j\_1]}\,\Gamma_{[ij\_1]} + 2\,u_{[ij\_1]}\,\Gamma_{[ij\_1]} + u_{[i1j]}\,(2\,\Gamma_{[ij]} - 3\,\Gamma xf_{[i1j\_1]}) +$$
$$3\,u_{[i1j\_1]}\,\Gamma xf_{[i1j\_1]} - 3\,u_{[ij\_1]}\,\Gamma xf_{[ij\_1]} + u_{[ij]}\,(-2\,\Gamma_{[ij]} + 3\,\Gamma xf_{[ij\_1]})))$$

In[52]:=
```
(* Check the derived numerical coefficients - the result have to be zero: *)
Simplify[IvSv - (aSv0 * v"[ij]" - (aSv1 * v"[i_1j]" + aSv2 * v"[i1j]" + aSv3 * v"[ij_1]" + aSv4 * v"[ij1]" + Svc))]
```

Out[52]= 0

In[53]:=
```
(**)
```

In[54]:=
```
(* All tesms are moved on left hand side to deduce the coefficients. *)
vExpresion = FullSimplify[
  (IvdρvdT + IvdρuvdX + IvdρvvdY + IvdpdY - (IvdΓdvdx2 + IvdΓdvdy2 + Ivρgy)) - IvSv]
```

Out[54]=
$$\frac{1}{6}\Big(-3\,(\max(0, Fx_{[ij]}) + \max(0, Fx_{[ij\_1]}) + 2\,Dvx_{[ij]})\,v_{[i\_1j]} -$$
$$3\,(\max(0, -Fx_{[i1j]}) + \max(0, -Fx_{[i1j\_1]}) + 2\,Dvx_{[i1j]})\,v_{[i1j]} +$$
$$3\,(2\max(0, F1y_{[ij]}) + 2\max(0, -F1y_{[ij\_1]}) + \max(0, Fx_{[i1j]}) + \max(0, Fx_{[i1j\_1]}) + \max(0, -Fx_{[ij]}) +$$
$$\max(0, -Fx_{[ij\_1]}) + 2\,Dvx_{[i1j]})\,v_{[ij]} + 6\,Dvx_{[ij]}\,v_{[ij]} + 8\,Dvy_{[ij]}\,v_{[ij]} + 8\,Dvy_{[ij1]}\,v_{[ij]} -$$
$$6\max(0, F1y_{[ij\_1]})\,v_{[ij\_1]} - 8\,Dvy_{[ij]}\,v_{[ij\_1]} - 6\max(0, -F1y_{[ij]})\,v_{[ij1]} - 8\,Dvy_{[ij1]}\,v_{[ij1]} -$$
$$\frac{1}{ht}\,3\,hx_{[i]}\,(2\,A\,ht\,p_{[ij]} - 2\,A\,ht\,p_{[ij\_1]} + (hy_{[j]}\,rho_{[ij]} + hy_{[j\_1]}\,rho_{[ij\_1]})\,(gy\,ht - v_{[ij]}) +$$
$$(hy_{[j]}\,rhopr_{[ij]} + hy_{[j\_1]}\,rhopr_{[ij\_1]})\,vpr_{[ij]}) + 4\,B\,((u_{[i1j]} - u_{[ij]})\,\Gamma_{[ij]} + (-u_{[i1j\_1]} + u_{[ij\_1]})\,\Gamma_{[ij\_1]}) +$$
$$\frac{1}{hy_{[j]} + hy_{[j\_1]}}\,6\,B\,(hy_{[j]}\,((-u_{[i1j]} + u_{[i1j\_1]})\,\Gamma xf_{[i1j]} + (u_{[ij]} - u_{[ij\_1]})\,\Gamma xf_{[ij]}) +$$
$$hy_{[j\_1]}\,((-u_{[i1j]} + u_{[i1j\_1]})\,\Gamma xf_{[i1j\_1]} + (u_{[ij]} - u_{[ij\_1]})\,\Gamma xf_{[ij\_1]}))\Big)$$

In[55]:=
```
(* Derive numerical coefficients *)
av0 = Simplify[Coefficient[vExpresion, v_"[ij]"]]
```

Out[55]=
$$\frac{1}{6}\left(6\max(0,F1y_{[i j]})+6\max(0,-F1y_{[i j\_1]})+3\max(0,Fx_{[i1 j]})+3\max(0,Fx_{[i1 j\_1]})+3\max(0,-Fx_{[i j]})+3\max(0,-Fx_{[i j\_1]})+6\,Dvx_{[i1 j]}+6\,Dvx_{[i j]}+8\,Dvy_{[i j]}+8\,Dvy_{[i j1]}+\frac{3\,hx_{[i]}\,hy_{[j]}\,rho_{[i j]}}{ht}+\frac{3\,hx_{[i]}\,hy_{[j\_1]}\,rho_{[i j\_1]}}{ht}\right)$$

In[56]:=
```
av1 = Simplify[-Coefficient[vExpresion, v_"[i_1j]"]]
```

Out[56]=
$$\frac{1}{2}\left(\max(0,Fx_{[i j]})+\max(0,Fx_{[i j\_1]})+2\,Dvx_{[i j]}\right)$$

In[57]:=
```
av2 = Simplify[-Coefficient[vExpresion, v_"[i1j]"]]
```

Out[57]=
$$\frac{1}{2}\left(\max(0,-Fx_{[i1 j]})+\max(0,-Fx_{[i1 j\_1]})+2\,Dvx_{[i1 j]}\right)$$

In[58]:=
```
av3 = Simplify[-Coefficient[vExpresion, v_"[ij_1]"]]
```

Out[58]=
$$\max(0,F1y_{[i j\_1]})+\frac{4\,Dvy_{[i j]}}{3}$$

In[59]:=
```
av4 = Simplify[-Coefficient[vExpresion, v_"[ij1]"]]
```

Out[59]=
$$\max(0,-F1y_{[i j]})+\frac{4\,Dvy_{[i j1]}}{3}$$

In[60]:=
```
bv =
 Simplify[-(vExpresion - (av0 * v_"[ij]" - (av1 * v_"[i_1j]" + av2 * v_"[i1j]" + av3 * v_"[ij_1]" + av4 * v_"[ij1]"))))]
```

Out[60]=
$$\frac{1}{2\,ht}hx_{[i]}\left(2\,A\,ht\,p_{[i j]}-2\,A\,ht\,p_{[i j\_1]}+gy\,ht\,hy_{[j]}\,rho_{[i j]}+gy\,ht\,hy_{[j\_1]}\,rho_{[i j\_1]}+hy_{[j]}\,rhopr_{[i j]}\,vpr_{[i j]}+hy_{[j\_1]}\,rhopr_{[i j\_1]}\,vpr_{[i j]}\right)+\frac{1}{3\left(hy_{[j]}+hy_{[j\_1]}\right)}B\left(hy_{[j]}\left(2\,u_{[i1 j\_1]}\,\Gamma_{[i j\_1]}-2\,u_{[i j\_1]}\,\Gamma_{[i j\_1]}-3\,u_{[i1 j\_1]}\,\Gamma xf_{[i1 j]}+u_{[i1 j]}\left(-2\,\Gamma_{[i j]}+3\,\Gamma xf_{[i1 j]}\right)+u_{[i j]}\left(2\,\Gamma_{[i j]}-3\,\Gamma xf_{[i j]}\right)+3\,u_{[i j\_1]}\,\Gamma xf_{[i j]}\right)+hy_{[j\_1]}\left(2\,u_{[i1 j\_1]}\,\Gamma_{[i j\_1]}-2\,u_{[i j\_1]}\,\Gamma_{[i j\_1]}-3\,u_{[i1 j\_1]}\,\Gamma xf_{[i1 j\_1]}+u_{[i1 j]}\left(-2\,\Gamma_{[i j]}+3\,\Gamma xf_{[i1 j\_1]}\right)+u_{[i j]}\left(2\,\Gamma_{[i j]}-3\,\Gamma xf_{[i j\_1]}\right)+3\,u_{[i j\_1]}\,\Gamma xf_{[i j\_1]}\right)\right)$$

In[61]:=
```
(* Check the derived numerical coefficients - the result have to be zero: *)
Simplify[vExpresion - (av0 * v_"[ij]" - (av1 * v_"[i_1j]" + av2 * v_"[i1j]" + av3 * v_"[ij_1]" + av4 * v_"[ij1]" + bv))]
```

Out[61]= 0

In[62]:=
```
(**)
```

In[63]:=
```
(* Derive the pressure equation
  The Pressure equation is derived after integration of conservation of
  mass equation over control volume of field variables and substitution of
  velocities. It is multiplied to time step. This make algorithm more stable,
 when are used small time steps for calculation of supersonic fluid flow.
    Integrated equation for conservation of mass:
    ∂tρ*hx"[i]"*hy"[j]"+(rhou"[i1j]"*u"[i1j]"-rhou"[ij]"*u"[ij]")*hy"[j]"+
     (rhov"[ij1]"*v"[ij1]"-rhov"[ij]"*v"[ij]")*hx"[i]"=0

    Substutude in integrated conservation
    of mass equation the velocities using preudo velocities:
     u"[ij]"=upseudo"[ij]"-du"[ij]"*(p"[ij]"-p"[i_1j]")
       v"[ij]"=vpseudo"[ij]"-dv"[ij]"*(p"[ij]"-p"[ij_1]")
 *)
```

In[64]:=
```
(* In unsteady term the density have to be substututed with
  pressure using eqation of state. At this way the numerical equation
  for pressure satisfy the sufficient condition for convergence of
  iterative method and under relaxation coefficients are not needed: *)
```

$$\text{Ipdrhodt} = \text{Simplify}\left[\left(\frac{p_{\text{"[ij]"}}}{\text{Temper}_{\text{"[ij]"}}} - \frac{\text{ppr}_{\text{"[ij]"}}}{\text{Temperpr}_{\text{"[ij]"}}}\right) * \text{hx}_{\text{"[i]"}} * \text{hy}_{\text{"[j]"}}\right];$$

In[65]:= $\text{Ipdrhoudx} = \text{Simplify}\big[(\text{rhou}_{\text{"[i1j]"}} * (\text{upseudo}_{\text{"[i1j]"}} - \text{du}_{\text{"[i1j]"}} * (p_{\text{"[i1j]"}} - p_{\text{"[ij]"}})) - \text{rhou}_{\text{"[ij]"}} * (\text{upseudo}_{\text{"[ij]"}} - \text{du}_{\text{"[ij]"}} * (p_{\text{"[ij]"}} - p_{\text{"[i\_1j]"}}))) * \text{hy}_{\text{"[j]"}} * \text{ht}\big];$

In[66]:= $\text{Ipdrhovdy} = \text{Simplify}\big[(\text{rhov}_{\text{"[ij1]"}} * (\text{vpseudo}_{\text{"[ij1]"}} - \text{dv}_{\text{"[ij1]"}} * (p_{\text{"[ij1]"}} - p_{\text{"[ij]"}})) - \text{rhov}_{\text{"[ij]"}} * (\text{vpseudo}_{\text{"[ij]"}} - \text{dv}_{\text{"[ij]"}} * (p_{\text{"[ij]"}} - p_{\text{"[ij\_1]"}}))) * \text{hx}_{\text{"[i]"}} * \text{ht}\big];$

In[67]:= pExpresion = FullSimplify[(Ipdrhodt + Ipdrhoudx + Ipdrhovdy)]

Out[67]=
$$\text{hx}_{[i]}\,\text{hy}_{[j]}\left(\frac{p_{[ij]}}{\text{Temper}_{[ij]}} - \frac{\text{ppr}_{[ij]}}{\text{Temperpr}_{[ij]}}\right) + \text{ht}\,\text{hy}_{[j]}\,(\text{rhou}_{[i1j]}\,(\text{du}_{[i1j]}\,(-p_{[i1j]} + p_{[ij]}) + \text{upseudo}_{[i1j]}) - \text{rhou}_{[ij]}\,(\text{du}_{[ij]}\,(p_{[i\_1j]} - p_{[ij]}) + \text{upseudo}_{[ij]})) + \text{ht}\,\text{hx}_{[i]}\,(-\text{rhov}_{[ij]}\,(\text{dv}_{[ij]}\,(-p_{[ij]} + p_{[ij\_1]}) + \text{vpseudo}_{[ij]}) + \text{rhov}_{[ij1]}\,(\text{dv}_{[ij1]}\,(p_{[ij]} - p_{[ij1]}) + \text{vpseudo}_{[ij1]}))$$

In[68]:=
```
(* Derive the numerical coefficients *)
```
ap0 = Simplify[Coefficient[pExpresion, $p_{\text{"[ij]"}}$]]

Out[68]=
$$\text{ht}\,\text{du}_{[i1j]}\,\text{hy}_{[j]}\,\text{rhou}_{[i1j]} + \text{ht}\,\text{du}_{[ij]}\,\text{hy}_{[j]}\,\text{rhou}_{[ij]} + \text{hx}_{[i]}\left(\text{ht}\,\text{dv}_{[ij]}\,\text{rhov}_{[ij]} + \text{ht}\,\text{dv}_{[ij1]}\,\text{rhov}_{[ij1]} + \frac{\text{hy}_{[j]}}{\text{Temper}_{[ij]}}\right)$$

In[69]:= ap1 = Simplify[-Coefficient[pExpresion, $p_{\text{"[i\_1j]"}}$]]

Out[69]= $\text{ht}\,\text{du}_{[ij]}\,\text{hy}_{[j]}\,\text{rhou}_{[ij]}$

In[70]:=
```
ap2 = Simplify[-Coefficient[pExpresion, p"[i1j]"]]
```
Out[70]= $ht\, du_{[i1j]}\, hy_{[j]}\, rhou_{[i1j]}$

In[71]:=
```
ap3 = Simplify[-Coefficient[pExpresion, p"[ij_1]"]]
```
Out[71]= $ht\, dv_{[ij]}\, hx_{[i]}\, rhov_{[ij]}$

In[72]:=
```
ap4 = Simplify[-Coefficient[pExpresion, p"[ij1]"]]
```
Out[72]= $ht\, dv_{[ij1]}\, hx_{[i]}\, rhov_{[ij1]}$

In[73]:=
```
bp =
  Simplify[-(pExpresion - (ap0 * p"[ij]" - (ap1 * p"[i_1j]" + ap2 * p"[i1j]" + ap3 * p"[ij_1]" + ap4 * p"[ij1]")))]
```
Out[73]= $ht\, hy_{[j]} \left(-rhou_{[i1j]}\, upseudo_{[i1j]} + rhou_{[ij]}\, upseudo_{[ij]}\right) + hx_{[i]} \left(\frac{hy_{[j]}\, ppr_{[ij]}}{Temperpr_{[ij]}} + ht\, rhov_{[ij]}\, vpseudo_{[ij]} - ht\, rhov_{[ij1]}\, vpseudo_{[ij1]}\right)$

In[74]:=
```
(* Check the derived numerical coefficients - the result have to be zero: *)
Simplify[
 -(pExpresion - (ap0 * p"[ij]" - (ap1 * p"[i_1j]" + ap2 * p"[i1j]" + ap3 * p"[ij_1]" + ap4 * p"[ij1]" + bp)))]
```
Out[74]= 0

In[75]:=
```
(**)
```

In[76]:=
```
(* Derive the energy equation *)
```

In[77]:=
```
(* Integration of unsteady term.
   It is multiplicated by time step to make numerical equation more stable,
when are used small time steps for calculation of supersonic fluid flows. *)
ITdrhoTdt = Simplify[(rho"[ij]" * Temper"[ij]" - rhopr"[ij]" * Temperpr"[ij]") * hx"[i]" * hy"[j]"];
```

In[78]:=
```
(* Integration of convective terms *)
```

In[79]:=
```
ITdrhouTdx = Simplify[("max(0,Fx"[i1j]")" * Temper"[ij]" - "max(0,-Fx"[i1j]")" * Temper"[i1j]" -
        "max(0,Fx"[ij]")" * Temper"[i_1j]" + "max(0,-Fx"[ij]")" * Temper"[ij]") * ht];
```

In[80]:=
```
ITdrhovTdy = Simplify[("max(0,Fy"[ij1]")" * Temper"[ij]" - "max(0,-Fy"[ij1]")" * Temper"[ij1]" -
        "max(0,Fy"[ij]")" * Temper"[ij_1]" + "max(0,-Fy"[ij]")" * Temper"[ij]") * ht];
```

In[81]:=
```
(* Integration of diffusion terms *)
```

In[82]:=
```
(*
DTx"[ij]"=CT1*Γ^λ"x_i^f"*hy"[j]"/(0.5*(hx"[i_1]"+hx"[i]"));

Γ^λ"x_i^f" is determined using average harmonic between two values:
    Γ^λ"x_i^f"=Hi(Γ^λ"[i_1j]",Γ^λ"[ij]",hx"[i_1]",hx"[i]");
*)
ITdGldTdx2 =
  Simplify[(DTx"[i1j]" * (Temper"[i1j]" - Temper"[ij]") - DTx"[ij]" * (Temper"[ij]" - Temper"[i_1j]")) * ht];
```

In[83]:=
```
(*
DTy"[ij]"=CT1*Γ^λ"y_j^f"*hx"[i]"/(0.5*(hy"[j_1]"+hy"[j]"));

Γ^λ"y_j^f" is determined using average harmonic between two values:
    Γ^λ"y_j^f"=Hi(Γ^λ"[ij_1]",Γ^λ"[ij]",hy"[j_1]",hy"[j]");
*)
ITdGldTdy2 =
  Simplify[(DTy"[ij1]" * (Temper"[ij1]" - Temper"[ij]") - DTy"[ij]" * (Temper"[ij]" - Temper"[ij_1]")) * ht];
```

In[84]:= (* Integrate source term *)

In[85]:= $ITdudx2 = \left(\frac{u_{\text{"[i1j]"}} - u_{\text{"[ij]"}}}{hx_{\text{"[i]"}}}\right)^2 * hx_{\text{"[i]"}} * hy_{\text{"[j]"}};$

In[86]:= $ITdvdy2 = \left(\frac{v_{\text{"[ij1]"}} - v_{\text{"[ij]"}}}{hy_{\text{"[j]"}}}\right)^2 * hx_{\text{"[i]"}} * hy_{\text{"[j]"}};$

In[87]:=
$$ITdvdxdudy2 = Simplify\left[\left(\left(\frac{v_{\text{"[i1j]"}} - v_{\text{"[ij]"}}}{\frac{1}{2} * (hx_{\text{"[i]"}} + hx_{\text{"[i1]"}})}\right)^2 + \left(\frac{v_{\text{"[ij]"}} - v_{\text{"[i\_1j]"}}}{\frac{1}{2} * (hx_{\text{"[i\_1]"}} + hx_{\text{"[i]"}})}\right)^2 + \left(\frac{v_{\text{"[i1j1]"}} - v_{\text{"[ij1]"}}}{\frac{1}{2} * (hx_{\text{"[i]"}} + hx_{\text{"[i1]"}})}\right)^2 + \left(\frac{v_{\text{"[i\_1j1]"}} - v_{\text{"[ij1]"}}}{\frac{1}{2} * (hx_{\text{"[i\_1]"}} + hx_{\text{"[i]"}})}\right)^2 + \left(\frac{u_{\text{"[ij1]"}} - u_{\text{"[ij]"}}}{\frac{1}{2} * (hy_{\text{"[j]"}} + hy_{\text{"[j1]"}})}\right)^2 + \left(\frac{u_{\text{"[ij]"}} - u_{\text{"[ij\_1]"}}}{\frac{1}{2} * (hy_{\text{"[j\_1]"}} + hy_{\text{"[j]"}})}\right)^2 + \left(\frac{u_{\text{"[i1j1]"}} - u_{\text{"[i1j]"}}}{\frac{1}{2} * (hy_{\text{"[j]"}} + hy_{\text{"[j1]"}})}\right)^2 + \left(\frac{u_{\text{"[i1j]"}} - u_{\text{"[i1j\_1]"}}}{\frac{1}{2} * (hy_{\text{"[j\_1]"}} + hy_{\text{"[j]"}})}\right)^2\right) * \frac{1}{2} * hx_{\text{"[i]"}} * \frac{1}{2} * hy_{\text{"[j]"}}\right];$$

In[88]:= $ITdudxdvdy2 = \left(\frac{u_{\text{"[i1j]"}} - u_{\text{"[ij]"}}}{hx_{\text{"[i]"}}} + \frac{v_{\text{"[ij1]"}} - v_{\text{"[ij]"}}}{hy_{\text{"[j]"}}}\right)^2 * hx_{\text{"[i]"}} * hy_{\text{"[j]"}};$

In[89]:= `ITΦ = Simplify[(2 * (ITdudx2 + ITdvdy2) + ITdvdxdudy2 - 2/3 * ITdudxdvdy2)]`

Out[89]=
$$hx_{[i]}\,hy_{[j]}\left(\frac{(u_{[i1j]}-u_{[i1j\_1]})^2}{(hy_{[j]}+hy_{[j\_1]})^2}+\frac{(u_{[i1j]}-u_{[i1j1]})^2}{(hy_{[j]}+hy_{[j1]})^2}+\frac{(u_{[ij]}-u_{[ij\_1]})^2}{(hy_{[j]}+hy_{[j\_1]})^2}+\frac{(u_{[ij]}-u_{[ij1]})^2}{(hy_{[j]}+hy_{[j1]})^2}+\frac{(v_{[i\_1j]}-v_{[ij]})^2}{(hx_{[i]}+hx_{[i\_1]})^2}+\frac{(v_{[i1j]}-v_{[ij]})^2}{(hx_{[i]}+hx_{[i1]})^2}+\frac{(v_{[i\_1j1]}-v_{[ij1]})^2}{(hx_{[i]}+hx_{[i\_1]})^2}+\frac{(v_{[i1j1]}-v_{[ij1]})^2}{(hx_{[i]}+hx_{[i1]})^2}\right)+2\left(\frac{hy_{[j]}\,(u_{[i1j]}-u_{[ij]})^2}{hx_{[i]}}+\frac{hx_{[i]}\,(v_{[ij]}-v_{[ij1]})^2}{hy_{[j]}}\right)-\frac{2}{3}\,hx_{[i]}\,hy_{[j]}\left(\frac{u_{[i1j]}-u_{[ij]}}{hx_{[i]}}+\frac{-v_{[ij]}+v_{[ij1]}}{hy_{[j]}}\right)^2$$

In[90]:= `ITdudx = (u"[i1j]" - u"[ij]") * hy"[j]";`

In[91]:= `ITdvdy = (v"[ij1]" - v"[ij]") * hx"[i]";`

In[92]:=
```
(*
The source term is: ST = CT2 Γ Φ + CT3 p(∂xu+∂xv)
     where:
      Φ=2((∂xu)^2+(∂yv)^2)+(∂xv+∂yu)^2-2/3(∂xu+∂yv)^2
 *)
ITST = CT2 * Γ * ITΦ + CT2 * p"[ij]" * (ITdudx + ITdvdy)
```

Out[92]=
$$CT2\,p_{[ij]}\left(hy_{[j]}\,(u_{[i1j]}-u_{[ij]})+hx_{[i]}\,(-v_{[ij]}+v_{[ij1]})\right)+CT2\,\Gamma\left(hx_{[i]}\,hy_{[j]}\left(\frac{(u_{[i1j]}-u_{[i1j\_1]})^2}{(hy_{[j]}+hy_{[j\_1]})^2}+\frac{(u_{[i1j]}-u_{[i1j1]})^2}{(hy_{[j]}+hy_{[j1]})^2}+\frac{(u_{[ij]}-u_{[ij\_1]})^2}{(hy_{[j]}+hy_{[j\_1]})^2}+\frac{(u_{[ij]}-u_{[ij1]})^2}{(hy_{[j]}+hy_{[j1]})^2}+\frac{(v_{[i\_1j]}-v_{[ij]})^2}{(hx_{[i]}+hx_{[i\_1]})^2}+\frac{(v_{[i1j]}-v_{[ij]})^2}{(hx_{[i]}+hx_{[i1]})^2}+\frac{(v_{[i\_1j1]}-v_{[ij1]})^2}{(hx_{[i]}+hx_{[i\_1]})^2}+\frac{(v_{[i1j1]}-v_{[ij1]})^2}{(hx_{[i]}+hx_{[i1]})^2}\right)+2\left(\frac{hy_{[j]}\,(u_{[i1j]}-u_{[ij]})^2}{hx_{[i]}}+\frac{hx_{[i]}\,(v_{[ij]}-v_{[ij1]})^2}{hy_{[j]}}\right)-\frac{2}{3}\,hx_{[i]}\,hy_{[j]}\left(\frac{u_{[i1j]}-u_{[ij]}}{hx_{[i]}}+\frac{-v_{[ij]}+v_{[ij1]}}{hy_{[j]}}\right)^2\right)$$

In[93]:= `(* Derive numerical coefficients for source term *)`

In[94]:= `aST0 = Simplify[-Coefficient[ITST, Temper"[ij]"]]`

Out[94]= 0

In[95]:= `aST1 = Simplify[Coefficient[ITST, Temper"[i_1j]"]]`

Out[95]= 0

In[96]:= `aST2 = Simplify[Coefficient[ITST, Temper"[i1j]"]]`

Out[96]= 0

In[97]:=
```
aST3 = Simplify[Coefficient[ITST, Temper"[ij_1]"]]
```

Out[97]= 0

In[98]:=
```
aST4 = Simplify[Coefficient[ITST, Temper"[ij1]"]]
```

Out[98]= 0

In[99]:=
```
(* STp = 0 => the coefficients aST0, aST1, aST2,
aST3 and aST4 are not checked for simplification. *)
```

In[100]:=
```
(* STp = 0 and all coefficients aST0, aST1, aST2, aST3 and aST4 are 0.
 Therefore STc is equal to the integrated source terms of energy equation. *)
STc = ITST
```

Out[100]=

$$\begin{aligned}
&CT2\, p_{[ij]} \left(hy_{[j]} \left(u_{[i1j]} - u_{[ij]}\right) + hx_{[i]} \left(-v_{[ij]} + v_{[ij1]}\right)\right) + \\
&CT2\, \Gamma \left(hx_{[i]}\, hy_{[j]} \left(\frac{\left(u_{[i1j]} - u_{[i1j_1]}\right)^2}{\left(hy_{[j]} + hy_{[j_1]}\right)^2} + \frac{\left(u_{[i1j]} - u_{[i1j1]}\right)^2}{\left(hy_{[j]} + hy_{[j1]}\right)^2} + \frac{\left(u_{[ij]} - u_{[ij_1]}\right)^2}{\left(hy_{[j]} + hy_{[j_1]}\right)^2} + \frac{\left(u_{[ij]} - u_{[ij1]}\right)^2}{\left(hy_{[j]} + hy_{[j1]}\right)^2} + \right.\right. \\
&\qquad \left. \frac{\left(v_{[i_1j]} - v_{[ij]}\right)^2}{\left(hx_{[i]} + hx_{[i_1]}\right)^2} + \frac{\left(v_{[i1j]} - v_{[ij]}\right)^2}{\left(hx_{[i]} + hx_{[i1]}\right)^2} + \frac{\left(v_{[i_1j1]} - v_{[ij1]}\right)^2}{\left(hx_{[i]} + hx_{[i_1]}\right)^2} + \frac{\left(v_{[i1j1]} - v_{[ij1]}\right)^2}{\left(hx_{[i]} + hx_{[i1]}\right)^2}\right) + \\
&\qquad \left. 2 \left(\frac{hy_{[j]} \left(u_{[i1j]} - u_{[ij]}\right)^2}{hx_{[i]}} + \frac{hx_{[i]} \left(v_{[ij]} - v_{[ij1]}\right)^2}{hy_{[j]}}\right) - \frac{2}{3}\, hx_{[i]}\, hy_{[j]} \left(\frac{u_{[i1j]} - u_{[ij]}}{hx_{[i]}} + \frac{-v_{[ij]} + v_{[ij1]}}{hy_{[j]}}\right)^2\right)
\end{aligned}$$

In[101]:=
```
(**)
```

In[102]:=
```
(* All terms are moved to the left hand side to derive the coefficients. *)
TExpresion = Simplify[(ITdrhoTdt + ITdrhouTdx + ITdrhovTdy - (ITdGldTdx2 + ITdGldTdy2)) - STc]
```

Out[102]=

$$-ht\left(DTx_{[ij]}\left(Temper_{[i\_1j]}-Temper_{[ij]}\right)+DTx_{[i1j]}\left(Temper_{[i1j]}-Temper_{[ij]}\right)\right)+ ht\left(-\max(0,Fx_{[ij]})\,Temper_{[i\_1j]}-\max(0,-Fx_{[i1j]})\,Temper_{[i1j]}+\left(\max(0,Fx_{[i1j]})+\max(0,-Fx_{[ij]})\right)Temper_{[ij]}\right)+ht\left(\left(\max(0,-Fy_{[ij]})+\max(0,Fy_{[ij1]})\right)Temper_{[ij]}-\max(0,Fy_{[ij]})\,Temper_{[ij\_1]}-\max(0,-Fy_{[ij1]})\,Temper_{[ij1]}\right)-ht\left(DTy_{[ij]}\left(-Temper_{[ij]}+Temper_{[ij\_1]}\right)+DTy_{[ij1]}\left(-Temper_{[ij]}+Temper_{[ij1]}\right)\right)+hx_{[i]}\,hy_{[j]}\left(rho_{[ij]}\,Temper_{[ij]}-rhopr_{[ij]}\,Temperpr_{[ij]}\right)-CT2\,p_{[ij]}\left(hy_{[j]}\left(u_{[i1j]}-u_{[ij]}\right)+hx_{[i]}\left(-v_{[ij]}+v_{[ij1]}\right)\right)-CT2\,\Gamma\left(hx_{[i]}\,hy_{[j]}\left(\frac{\left(u_{[i1j]}-u_{[i1j\_1]}\right)^2}{\left(hy_{[j]}+hy_{[j\_1]}\right)^2}+\frac{\left(u_{[i1j]}-u_{[i1j1]}\right)^2}{\left(hy_{[j]}+hy_{[j1]}\right)^2}+\frac{\left(u_{[ij]}-u_{[ij\_1]}\right)^2}{\left(hy_{[j]}+hy_{[j\_1]}\right)^2}+\frac{\left(u_{[ij]}-u_{[ij1]}\right)^2}{\left(hy_{[j]}+hy_{[j1]}\right)^2}+\frac{\left(v_{[i\_1j]}-v_{[ij]}\right)^2}{\left(hx_{[i]}+hx_{[i\_1]}\right)^2}+\frac{\left(v_{[i1j]}-v_{[ij]}\right)^2}{\left(hx_{[i]}+hx_{[i1]}\right)^2}+\frac{\left(v_{[i\_1j1]}-v_{[ij1]}\right)^2}{\left(hx_{[i]}+hx_{[i\_1]}\right)^2}+\frac{\left(v_{[i1j1]}-v_{[ij1]}\right)^2}{\left(hx_{[i]}+hx_{[i1]}\right)^2}\right)+2\left(\frac{hy_{[j]}\left(u_{[i1j]}-u_{[ij]}\right)^2}{hx_{[i]}}+\frac{hx_{[i]}\left(v_{[ij]}-v_{[ij1]}\right)^2}{hy_{[j]}}\right)-\frac{2}{3}\,hx_{[i]}\,hy_{[j]}\left(\frac{u_{[i1j]}-u_{[ij]}}{hx_{[i]}}+\frac{-v_{[ij]}+v_{[ij1]}}{hy_{[j]}}\right)^2\right)$$

In[103]:=
```
(* Derive numerical coefficients *)
aT0 = Simplify[Coefficient[TExpresion, Temper"[ij]"]]
```

Out[103]=

$$\max(0,Fx_{[i1j]})\,ht+\max(0,-Fx_{[ij]})\,ht+\max(0,-Fy_{[ij]})\,ht+\max(0,Fy_{[ij1]})\,ht+ht\,DTx_{[i1j]}+ht\,DTx_{[ij]}+ht\,DTy_{[ij]}+ht\,DTy_{[ij1]}+hx_{[i]}\,hy_{[j]}\,rho_{[ij]}$$

In[104]:=
```
aT1 = Simplify[-Coefficient[TExpresion, Temper"[i_1j]"]]
```

Out[104]=

$$ht\left(\max(0,Fx_{[ij]})+DTx_{[ij]}\right)$$

In[105]:=
```
aT2 = Simplify[-Coefficient[TExpresion, Temper"[i1j]"]]
```

Out[105]=

$$ht\left(\max(0,-Fx_{[i1j]})+DTx_{[i1j]}\right)$$

In[106]:=
```
aT3 = Simplify[-Coefficient[TExpresion, Temper"[ij_1]"]]
```

Out[106]=

$$ht\left(\max(0,Fy_{[ij]})+DTy_{[ij]}\right)$$

In[107]:=
```
aT4 = Simplify[-Coefficient[TExpresion, Temper"[ij1]"]]
```

Out[107]=

$$ht\left(\max(0,-Fy_{[ij1]})+DTy_{[ij1]}\right)$$

In[108]:=
```
bT = Simplify[-(TExpresion - (aT0 * Temper"[ij]" -
        (aT1 * Temper"[i_1j]" + aT2 * Temper"[i1j]" + aT3 * Temper"[ij_1]" + aT4 * Temper"[ij1]" + STc)))]
```

Out[108]=

$$hx_{[i]}\,hy_{[j]}\,rhopr_{[ij]}\,Temperpr_{[ij]}$$

In[109]:=
```
(* Check derived numerical coefficients - the result have to be zero: *)
Simplify[TExpresion - (aT0 * Temper"[ij]" -
    (aT1 * Temper"[i_1j]" + aT2 * Temper"[i1j]" + aT3 * Temper"[ij_1]" + aT4 * Temper"[ij1]" + bT + STc))]
```

Out[109]= 0